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The Fifteen-Parameter Conformal Group

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Abstract

The space of lines in a Hermitean quadric of signature (2, 2) in complex projective three-space is a quadric of signature (2, 4) in real projective five-space, the conformal compactification of Minkowski space. This geometric fact leads to the classical isomorphism of $PSU(2, 2)$ and the identity component of $PO(2, 4; \mathbb{R})$, the 15-parameter conformal group. In this paper it is shown how the geometry and the isomorphism, for all components of $PO(2, 4; \mathbb{R})$, arise naturally from a real form of the Clifford algebra, and its associated spin groups, of a certain complex vector space determined by skew-symmetric 4 x 4 matrices and having their Pfaffian as quadratic form.

1. *Introduction*

1.1. The group of conformal transformations of a real vector space V, having a quadratic form with signature $(s, n - s)$, s minus signs and $n - s$ plus signs, has been known for over a century; it is generated by inversions in spheres analogous to Möbius inversions of Euclidean space. By adjoining a cone at infinity to V, the conformal compactification Ψ of V is obtained; it is a quadric of signature $(s + 1, n - s + 1)$, in real projective space, on which the conformal group acts globally as the projective orthogonal group $PO(s + 1, n - s + 1; \mathbb{R})$. When V is Minkowski space, Ψ has signature (2, 4) and $PO(2, 4; \mathbb{R})$ is the 15-parameter conformal group.

The 15-parameter conformal group appeared, just after the turn of the century, as the group of transformations preserving the free-field Maxwell equations; since then it has played a role in physics, but in this context is usually described in terms of $PSU(2, 2)$, which is isomorphic to the identity component of $PO(2, 4; \mathbb{R})$. Such a description forms the basis of Penrose's twistors (Penrose, 1967).

It was observed by E. Study, E. Cartan, and others, early in this century, that the complex lines lying in a Hermitean quadric Φ of signature (2, 2) in complex projective 3-space are parametrized by the points of a quadric of signature (2,4) in real projective 5-space (Study, 1924, p. 235). This is the geometric reason for the isomorphism of $PSU(2, 2)$ with the identity component of $PO(2, 4; \mathbb{R})$. If compactified Minkowski space Ψ is viewed as the space of generators of Φ, the collineations and correlations of projective space which preserve Φ give, by permuting the generators of Φ, the conformal transformations of Ψ.

However, in this beautiful viewpoint for the conformal geometry of Minkowski space, it is overlooked that this is a natural consequence of an explicit description of a real form of a certain Clifford algebra. In this paper we describe this natural algebraic setting and obtain from it both the geometry and the isomorphism. An overview of this development follows.

1.2. A cetain six-dimensional complex vector space V of 8 x 8 matrices, determined by skew-symmetric 4 x 4 matrices and having their Pfaffian as quadratic form, has all complex 8 x 8 matrices as its Clifford algebra. The complex four-dimensional Plücker quadric Ω, defined by the vanishing of the Pfaffian and lying in the complex projective 5-space P^5 based on V, describes lines in complex P^3. The Clifford group of V contains a subgroup, with identity component isomorphic to $SL(4; \mathbb{C})$, which maps, under the homomorphism to $PO(V)$, onto the group of collineations of P^5 preserving Ω, and which corresponds to collineations and correlations of P^3 as they operate on lines.

A Hermitean form of signature (2, 2) defines a real 5-dimensional Hermitean quadric Φ in P^3, and the antipolarity of P^3 with respect to Φ arises from a complex conjugation of V and its Clifford algebra. The points of Ω that represent generators of Φ are the points of Ω fixed by the antipolarity as it operates on lines and turns out to be compactified Minkowski space Ψ. The fixed points of the complex conjugation on V give a real form V_0 of V; Ψ is a quadric of signature (2, 4) lying in the real five-dimensional projective space based on V_0. The spin group Spin (V_0), with identity component isomorphic to $SU(2, 2)$, maps onto $PO(V_0)$, isomorphic to $PO(2, 4; \mathbb{R})$, the group of collineations of real projective 5-space preserving Ψ, and corresponds to collineations and correlations of complex P^3 which, as they operate on lines, commute with the antipolarity.

The homomorphism from Spin(V_0) onto $PO(V_0)$ extends to all four components the usual homomorphism from $SU(2, 2)$ to $PO(2, 4; \mathbb{R})$. The collineations and correlations of P^3 which commute with the antipolarity preserve Φ and permute its generators; this corresponds to $PO(V_0)$ operating on Ψ by conformal transformations.

1.3. Sections 2 and 3 of this paper are summaries of facts needed later about conformal geometry and about spin groups and Minkowski space. The Clifford algebra arising from skew-symmetric matrices is developed in Section 4 and line geometry is described in terms of it in Section 5. Finally, in Section 6, the isomorphism, in its geometric form, is described in terms of spin groups.

2. *Conformal Geometry*

2.1. Let V be a real vector space of dimension n having a nondegenerate quadratic form $(x|x)$ of arbitrary signature. A sphere of V with center a and square-radius R is described by $(x - a|x - a) = R$. A sphere may not have any real points in V, but always has points in the complexification of V. A sphere is nonsingular if $R \neq 0$ and singular, or a cone, if $R = 0$. As a limiting case of spheres with center $a + tu$ and square-radius $t^2(u|u)$, $t \to \pm\infty$, we have the hyperplane $(u|x - a) = 0$, which is nonsingular if $(u|u) \neq 0$ and singular if $(u|u) = 0$.

2.2. Inversion in the nonsingular sphere $(x - a|x - a) = R$ is the transformation σ from V to itself which sends x in V to the point $\sigma(x)$ collinear with a and x and satisfying $(\sigma(x) - a|x - a) = R$. One has

$$\sigma(x) = a + \frac{R}{(x - a|x - a)}(x - a)$$

This is defined only for $(x - a|x - a) \neq 0$, so inversion is an involutive Cremona transformation of V. The limiting case of inversions in the spheres with center $a + tu$ and square-radius $t^2(u|u)$, $t \to \pm\infty$, $(u|u) \neq 0$, is the reflection

$$x \to x - \frac{2(u|x - a)}{(u|u)}u$$

in the nonsingular hyperplane $(u|x - a) = 0$.

The inversion σ, above, is the composition of translations, a homothety, and the inversion $x \to [1/(x|x)]x$ in the unit sphere $(x|x) = 1$ centered at the origin. The effect of a general inversion may be ascertained from that of $x \to [\alpha/(x|x)]x$ or $x \to [1/(x|x)]x$, inversion in $(x|x) = \alpha$ or $(x|x) = 1$.

Under inversion, a sphere is transformed into a sphere or hyperplane according as the sphere does not or does pass through the center of the sphere of inversion, nonsingular or singular according as the sphere is nonsingular or singular. The same assertion holds for the transform of a hyperplane.

2.3. The vector space V has a nondegenerate pseudo-Riemannian metric given by its quadratic form: $ds^2 = (dx|dx)$. In this metric, inversions in nonsingular spheres are conformal transformations and reflections in nonsingular hyperplanes are isometries.

All inversions in nonsingular spheres and reflections in nonsingular hyperplanes generate a group of Cremona transformations of V, the Möbius group $M(V)$. A reflection in a nonsingular hyperplane can be obtained as a product of inversions in nonsingular spheres, so inversions suffice to generate $M(V)$. Transformations in $M(V)$ are conformal transformations of V; by a theorem of Liouville and Lie, $M(V)$ constitutes all such transformations.

Reflections in nonsingular hyperplanes passing through the origin generate the orthogonal group $O(V)$; $O(V)$ is a subgroup of $M(V)$. Homotheties of V can be obtained as products of inversions in concentric nonsingular spheres, translations of V as products of reflections in parallel nonsingular hyperplanes. Consequently, $M(V)$ contains all transformations

$$x \to \lambda ux + a, \qquad \lambda \neq 0 \text{ real}, \qquad u \text{ in } O(V), \qquad a \text{ in } V$$

Furthermore, $M(V)$ is generated by these transformations and the single additional inversion $x \to [1/(x|x)]x$.

2.4. With V as in 2.1, let $V^+ = V + \mathbb{R}e$ consist of vectors $x' = x + ye$, x in V, y real. Extend the quadratic form of V to V^+ by $(x'|x') = (x|x) + y^2$; this extension is nondegenerate and its signature has one additional plus sign. Let Ψ_1 denote the unit sphere $(x'|x') = 1$ of V^+.

Stereographic projection is the map ψ which sends x in V to the point $\psi(x) \neq e$ in Ψ_1 in which the line joining x to e meets Ψ_1. One has

$$\psi(x) = \frac{2}{(x|x)+1}x + \frac{(x|x)-1}{(x|x)+1}e = e + \frac{2}{(x-e|x-e)}(x-e)$$

for x in V. ψ is a Cremona transformation, bijective from $V - \{(x|x) = -1\}$ to $\Psi_1 - \{y = 1\}$; $y = 1$ is the tangent hyperplane to Ψ_1 at e. ψ coincides with the restriction to V of inversion in the sphere $(x' - e|x' - e) = 2$ of V^+. Consequently, with respect to the metric $(dx'|dx') = (dx|dx) + dy^2$ of V^+ restricted to V and Ψ_1, stereographic projection is conformal.

2.5. The image under ψ of the sphere $(x - a|x - a) = R$ of V is the intersection of Ψ_1 with the hyperplane $(p|x') = 1$ or $(a - e|x') = 0$ of V^+ according as $(a|a) - R + 1$ is $\neq 0$ or $= 0$, where

$$p = \frac{2}{(a|a)-R+1}a + \frac{(a|a)-R-1}{(a|a)-R+1}e$$

When the sphere is nonsingular, this hyperplane is not tangent to Ψ_1; its pole with respect to Ψ_1, p or $a - e$, the latter at infinity, does not lie on Ψ_1.

The perspectivity of Ψ_1 from a point p of V^+ not on Ψ_1 is the map π which sends x' in Ψ_1 to the point $\pi(x')$ in Ψ_1, which is the second intersection of the line joining x' to p with Ψ_1. One has

$$\pi(x') = \frac{(p|p)-1}{1-2(p|x')+(p|p)}x' + \frac{2-2(p|x')}{1-2(p|x')+(p|p)}p$$

$$= p + \frac{(p|p)-1}{(x'-p|x'-p)}(x'-p)$$

π coincides with the restriction to Ψ_1 of inversion in the nonsingular sphere $(x'-p|x'-p) = (p|p) - 1$, orthogonal to Ψ_1, of V^+. π is an involutive Cremona transformation and is conformal. Likewise, the perspectivity of Ψ_1 along u in V^+, $(u|u) \neq 0$, or from the point u at infinity not on Ψ_1, coincides with the restriction to Ψ_1 of reflection in the nonsingular hyperplane $(u|x') = 0$ of V^+.

If σ is inversion in the nonsingular sphere $(x - a|x - a) = R$ of V, then σ transported to Ψ_1 by stereographic projection, $\psi\sigma\psi^{-1}$, coincides with the perspectivity of Ψ_1 from p or a–e, as above, according as $(a|a) - R + 1$ is $\neq 0$ or $= 0$. Thus, under stereographic projection from V to Ψ_1, to nonsingular spheres of V correspond the intersections of Ψ_1 with nontangent hyperplanes of V^+. To inversion in a nonsingular sphere of V corresponds the perspectivity of Ψ_1 from the point not on Ψ_1 that is the pole of the hyperplane corresponding to the sphere.

Consequently, $\sigma \to \psi\sigma\psi^{-1}$, with inverse $\pi \to \psi^{-1}\pi\psi$, is an isomorphism of $M(V)$ with the group of transformations of Ψ_1 generated by perspectivities of Ψ_1 from points not on Ψ_1.

2.6. With V as in 2.1, let $V^{\pm} = V + \mathbb{R}e + \mathbb{R}f$ consist of vectors $x'' = x + ye + zf$, x in V, y and z real. Extend the quadratic form of V to $V^{\pm}$ by $(x''|x'') = (x|x) + y^2 - z^2$; this extension is nondegenerate and its signature has additionally one plus sign and one minus sign. Let Ψ denote the quadric in the real projective space P^{n+1} based on $V^{\pm}$ which is the image under the canonical map from $V^{\pm}$ to P^{n+1} of the cone K, $(x''|x'') = 0$, of $V^{\pm}$. If H is a singular hyperplane of $V^{\pm}$ not passing through the origin, then $H \cap K$, with metric $(dx''|dx'') = (dx|dx) + dy^2 - dz^2$, is conformal to V. The collection of images in Ψ of such $H \cap K$ generate an atlas which gives Ψ a conformal structure.

An isometry of $V^{\pm}$ preserves K and hence induces a collineation of P^{n+1} which preserves Ψ. Such a map is a conformal transformation of Ψ, and hence $PO(V^{\pm}) = O(V^{\pm})/\{\pm 1\}$ is a group of conformal transformations of Ψ; it is in fact all of them. $PO(V^{\pm})$ is transitive on Ψ, so the compact quadric Ψ is homogeneous under its group of conformal automorphisms. $PO(V^{\pm})$ has real dimension $\frac{1}{2}(n + 2)(n + 1)$.

For p in $V^{\pm}$, $(p|p) \neq 0$, the map $x'' \to x'' - [2(p|x'')/(p|p)]p$ is an isometry of $V^{\pm}$, reflection in the nonsingular hyperplane $(p|x'') = 0$ passing through the origin. Such reflections generate $O(V^{\pm})$. A reflection, as above, corresponds to a collineation of P^{n+1} which, when restricted to Ψ, coincides with the perspectivity of Ψ from p: The image of x'' is the second intersection of the line joining x'' to p with Ψ. p is not on Ψ and is the pole of the hyperplane $(p|x'') = 0$ of P^{n+1}. Consequently: $PO(V^{\pm})$, as a transformation group of Ψ, is generated by perspectivities of Ψ from points not on Ψ.

The map $x \to x + \frac{1}{2}[(x|x) - 1]\,e + \frac{1}{2}[(x|x) + 1]\,f$ of V into $\{-y + z = 1\} \cap K$, followed by the canonical map from $V^{\pm}$ to P^{n+1}, gives a conformal inclusion of V into Ψ with image an open dense subset of Ψ. The complement of the image is the cone at infinity for V, described in Ψ by $-y + z = 0$. Ψ is the conformal compactification of V. Points of K are classical polyspherical coordinates for V. Viewed otherwise, $x' \to x' + f$ gives an inclusion of V^+ into P^{n+1} and hence of Ψ_1 into Ψ, this, preceded by stereographic projection ψ from V to Ψ_1, is the inclusion of V into Ψ. This latter viewpoint requires special treatment of points for which $(x|x) = -1$ or $z = 0$.

2.7. From 2.5, the image under the inclusion of V into Ψ of the sphere $(x - a|x - a) = R$ of V is the intersection of Ψ with the hyperplane $(p|x'') = 0$ of P^{n+1} having pole given by

$$p = a + \frac{(a|a) - R - 1}{2}\,e + \frac{(a|a) - R + 1}{2}\,f$$

in $V^{\pm}$. When the sphere is nonsingular, $(p|p) = R$ is $\neq 0$ and p does not lie on Ψ. To inversion in the sphere $(x - a|x - a) = R$ of V corresponds the perspectivity $x'' \to x'' - [2(p|x'')/(p|p)]p$ of Ψ from p. The former generate $M(V)$ and the latter generate $PO(V^{\pm})$; this leads to the isomorphism $M(V) \simeq PO(V^{\pm})$. $M(V)$ is a Lie group.

For the sphere $(x|x) = \alpha$ of V we have

$$p = \frac{-\alpha - 1}{2} e + \frac{-\alpha + 1}{2} f$$

To inversion in this sphere corresponds the perspectivity

$$\begin{aligned} x &\to x \\ e &\to \frac{1}{2}\left(-\alpha - \frac{1}{\alpha}\right)e + \frac{1}{2}\left(-\alpha + \frac{1}{\alpha}\right)f \\ f &\to \frac{1}{2}\left(\alpha - \frac{1}{\alpha}\right)e + \frac{1}{2}\left(\alpha + \frac{1}{\alpha}\right)f \end{aligned}$$

of Ψ.

2.8. Let V consist of real n-dimensional column vectors and have quadratic form

$$(x|x) = -(x^1)^2 - \cdots - (x^s)^2 + (x^{s+1})^2 + \cdots + (x^n)^2 = {}^t xAx$$

with matrix

$$A = \begin{pmatrix} -1_s & 0 \\ 0 & 1_{n-s} \end{pmatrix}$$

Then the quadratic form of $V^{\pm}$ has matrix

$$A^{\pm} = \left(\begin{array}{c|cc} A & \multicolumn{2}{c}{0} \\ \hline 0 & 1 & 0 \\ & 0 & -1 \end{array}\right)$$

Thus

$$M(V) \simeq PO(s + 1, n - s + 1; \mathbb{R}) = O(s + 1, n - s + 1; \mathbb{R})/\{\pm 1_{n+2}\}$$

where $O(s + 1, n - s + 1; \mathbb{R})$ consists of all $(n + 2) \times (n + 2)$ real matrices g satisfying ${}^t gA^{\pm}g = A^{\pm}$. Via this isomorphism, conformal transformations of V are represented by $g \bmod\{\pm 1_{n+2}\}$, with g in $O(s + 1, n - s + 1; \mathbb{R})$, as follows. Rotations:

$$x \to ux \qquad g = \left(\begin{array}{c|c} u & 0 \\ \hline 0 & \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \end{array}\right)$$

u in $O(V) = O(s, n-s; \mathbb{R})$. This is just $O(V)$ as a subgroup of $M(V) \simeq PO(V^{\pm})$.
Translations:

$$x \to x + a \quad a \text{ in } V \qquad g = \left(\begin{array}{c|cc} 1_n & -a & a \\ \hline {}^t aA & 1 - \frac{1}{2}(a|a) & \frac{1}{2}(a|a) \\ {}^t aA & -\frac{1}{2}(a|a) & 1 + \frac{1}{2}(a|a) \end{array}\right)$$

Homotheties:

$$x \to \lambda x \quad \lambda \neq 0 \quad \text{real} \qquad g = \left(\begin{array}{c|cc} 1_n & 0 & \\ \hline 0 & \frac{1}{2}(\lambda + 1/\lambda) & \frac{1}{2}(\lambda - 1/\lambda) \\ & \frac{1}{2}(\lambda - 1/\lambda) & \frac{1}{2}(\lambda + 1/\lambda) \end{array}\right)$$

Inversions:

$$x \to \frac{\alpha}{(x|x)} x \quad \text{c.f. } 2.7 \qquad g = \left(\begin{array}{c|cc} 1_n & 0 & \\ \hline 0 & \frac{1}{2}(-\alpha - 1/\alpha) & \frac{1}{2}(\alpha - 1/\alpha) \\ & \frac{1}{2}(-\alpha + 1/\alpha) & \frac{1}{2}(\alpha + 1/\alpha) \end{array}\right)$$

When $s = 0$, V is n-dimensional Euclidean space, Ψ is an n-dimensional sphere, and $M(V) \simeq PO(1, n+1; \mathbb{R})$ is the familiar two-component Möbius group. When $n = 4$ and $s = 1$, V is Minkowski space, Ψ is its conformal compactification, and $M(V) \simeq PO(2, 4; \mathbb{R})$ is the four-component 15-parameter conformal group.

3. *Spin Groups*

3.1. Let V be a complex vector space of even dimension n having a nondegenerate quadratic form $(x|x)$. The Clifford algebra $C(V)$, of complex dimension 2^n, contains V and is generated by V subject to the relations $xy + yx = -2(x|y) \cdot 1$, x and y in V. $C(V)$ is universal in the sense that any linear map φ of V into an algebra satisfying $\varphi(x)\varphi(y) + \varphi(y)\varphi(x) = -2(x|y) \cdot 1$ in the algebra extends uniquely to an algebra homomorphism of $C(V)$ into the algebra. Since the dimension of V is even, $C(V)$ is a simple algebra and this extension is either injective or trivial. As a consequence of this universality, $C(V)$ possesses an involution $a \to {}^I a$ and an anti-involution $a \to {}^J a$ which are extensions of ${}^I x = -x$ and ${}^J x = x$, x in V, respectively. An element a of $C(V)$ is even or odd as ${}^I a$ is a or $-a$.

The invertible elements g of $C(V)$ for which $g V^I g^{-1} = V$ constitute the Clifford group $D(V)$. For g in $D(V)$ the linear transformation $p(g)$ of V defined by $p(g)x = g x^I g^{-1}$ is in the orthogonal group $O(V)$. For nonisotropic a in V, $(a|a) \neq 0$,

$$p(a)x = x - 2\frac{(x|a)}{(a|a)} a$$

is reflection in the nonsingular hyperplane of V which is orthogonal to a and passes through the origin. Such reflections generate $O(V)$, so p maps $D(V)$ homomorphically onto $O(V)$; the kernel of p is $\mathbb{C}^* \cdot 1$ consisting of $\lambda \cdot 1$, $\lambda \neq 0$ complex. An element g of $D(V)$ has image in the special orthogonal group $O^+(V)$ or its coset in $O(V)$ as g is even or odd. p maps the subgroup $D^+(V)$ of $D(V)$ consisting of even elements onto $O^+(V)$ with kernel $\mathbb{C}^* \cdot 1$. Since n is even, the image of $z = e_1 e_2 \ldots e_n$ in $D^+(V)$, with the e_i orthonormal, $(e_i|e_i) = \pm 1$, is $p(z) = -1$ in $O^+(V)$.

Let W be a nonsingular subspace of V of even dimension; that is, the restriction to W of the quadratic form of V is nondegenerate. Then $C(W)$ is a subalgebra of $C(V)$. Any x in V is uniquely $x = y + y'$ with y in W and y' orthogonal to W; for h in $D(W)$, $hy' = y'^I h$, so $hx^I h^{-1} = hy^I h^{-1} + y'$. Hence $D(W)$ is a subgroup of $D(V)$ and its inclusion corresponds, under p, to the inclusion of $O(W)$ in $O(V)$ obtained by sending v to the linear transformation $y + y' \to vy + y'$.

$$\begin{array}{ccc} D(W) & \xrightarrow{\text{incl.}} & D(V) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ O(W) & \xrightarrow[\text{incl.}]{} & O(V) \end{array}$$

The same assertions hold for the subgroups of even elements and the special orthogonal groups.

3.2. Let V_0 be a real vector space of even dimension n having a nondegenerate quadratic form $(x|x)$ of arbitrary signature. The considerations of 3.1 hold for V_0 with the evident changes. p now has kernel $\mathbb{R}^* \cdot 1$.

For g in $D(V_0)$, define the scalar Ng by $g^{IJ}g = g^{JI}g = Ng \cdot 1$; $g \to Ng$ is multiplicative. Define the spin groups, subgroups of $D(V_0)$, by

$\mathrm{Spin}(V_0)$	$Ng = \pm 1$
$\mathrm{Spin}^+(V_0)$	$Ng = \pm 1$ and g even
$\mathrm{Spin}^1(V_0)$	$Ng = +1$ and g even

$\mathrm{Spin}^1(V_0)$ is connected and is the identity component of the other groups. p maps these groups onto $O(V_0)$, $O^+(V_0)$, and their identity component $O^1(V_0)$, respectively. The kernel is $\{\pm 1\}$ in each case. For if V_0 contains a vector e_- such that $Ne_- = -e_-^2 = (e_-|e_-) = -1$, then $D(V_0) = \mathrm{Spin}(V_0) \cdot \mathbb{R}^*$ and $D^+(V_0) = \mathrm{Spin}^+(V_0) \cdot \mathbb{R}^* = \mathrm{Spin}^1(V_0) \cdot \{1, e_-\} \cdot \mathbb{R}^*$; if the quadratic form on V_0 is positive definite so $Ng > 0$ always, then $\mathrm{Spin}^1(V_0)$ coincides with $\mathrm{Spin}^+(V_0)$ as does $O^1(V_0)$ with $O^+(V_0)$. Since n is even, ${}^Iz = z$, ${}^Jz = (-1)^{n(n-1)/2}z$, $z^2 = (-1)^{n(n-1)/2}\Delta$, and $Nz = \Delta$, where $\Delta = \Pi(e_i|e_i)$. z in $\mathrm{Spin}^+(V_0)$ lies in $\mathrm{Spin}^1(V_0)$ when $\Delta = +1$.

Let W_0 be a nonsingular subspace of V_0 of even dimension. The inclusion of $D(W_0)$ in $D(V_0)$ restricts to the inclusion of $\mathrm{Spin}(W_0)$ in $\mathrm{Spin}(V_0)$ and this corresponds, under p, to the inclusion of $O(W_0)$ in $O(V_0)$. The same assertions hold for the other two spin groups.

3.3. Let V be a complex vector space as in Section 3.1 and let $x \to {}^c x$ be a complex conjugation on V for which the quadratic form is real: $x \to {}^c x$ is real linear, ${}^c(\lambda x) = \bar{\lambda}\, {}^c x$, ${}^c({}^c x) = x$, and $({}^c x \mid {}^c y) = \overline{(x \mid y)}$. Let V_0 be the real subspace of real vectors: ${}^c x = x$. Note that any x in V for which ${}^c x = \lambda x$, λ complex, is necessarily of the form $x = \mu x_0$, x_0 in V_0 and $|\mu| = 1$. For ${}^c({}^c x) = x$ yields $\bar{\lambda}\lambda = 1$, and then set $\lambda = \mu^{-2}$ and $x_0 = \mu^{-1} x$. This observation will be applied to various spaces.

The conjugation on V extends to one on $C(V)$, $a \to {}^c a$. Since V_0 generates $C(V)$ and $C(V_0)$ as complex and real algebras, respectively, $C(V_0)$ is the real subalgebra of $C(V)$ consisting of real elements: ${}^c a = a$. The conjugation on $C(V)$ sends $D(V)$ to itself; $D(V_0)$ in $D(V)$ is the subgroup of real elements: ${}^c g = g$. The conjugation on V induces one on linear transformations of V by ${}^c u x = {}^c(u\, {}^c x)$. This conjugation sends $O(V)$ to itself; $O(V_0)$ in $O(V)$ is the subgroup of real elements: ${}^c u = u$. Note that ${}^c(p(g)) = p({}^c g)$, so p sends $D(V_0)$ onto $O(V_0)$.

Since ${}^c(p(g)) = p({}^c g)$, for $p(g)$, g in $D(V)$, to be real, ${}^c g g^{-1}$ must be in the kernel of p: ${}^c g = \lambda g$. Hence, the preimage under p of $O(V_0)$ in $D(V)$ is $D(V_0) \cdot \{\mu 1\} = \mathrm{Spin}(V_0) \cdot \mathbb{C}^*$, $|\mu| = 1$. Likewise, the preimages under p of $O^+(V_0)$ and $O^1(V_0)$ are $D^+(V_0) \cdot \{\mu 1\} = \mathrm{Spin}^+(V_0) \cdot \mathbb{C}^*$ and $\mathrm{Spin}^1(V_0) \cdot \mathbb{C}^*$; these lie in $D^+(V)$.

3.4. Let V be a complex vector space of even dimension as in Section 3.1. The projective orthogonal group of V is $PO(V) = O(V)/\{\pm 1\}$. $g \to p(g)$ mod $\{\pm 1\}$ maps $D(V)$ homomorphically onto $PO(V)$ with kernel $\mathbb{C}^* \cdot \{1, z\}$. If W is a nonsingular subspace of V of even dimension, the inclusion of $D(W)$ in $D(V)$ corresponds to the inclusion of $O(W)$ in $PO(V)$ obtained by sending v into its image in $O(V)$ modulo $\{\pm 1\}$. The same assertions apply to the projective special orthogonal group $PO^+(V) = O^+(V)/\{\pm 1\}$ since V has even dimension and z lies in $D^+(V)$.

Let V_0 be a real vector space of even dimension as in Section 3.2. $g \to p(g)$ mod $\{\pm 1\}$ maps $\mathrm{Spin}(V_0)$ onto $PO(V_0) = O(V_0)/\{\pm 1\}$ with kernel $\{\pm 1, \pm z\}$; likewise $\mathrm{Spin}^+(V_0)$ and $PO^+(V_0) = O^+(V_0)/\{\pm 1\}$. If the number of e_i for which $(e_i \mid e_i) = -1$ is even, so that $Nz = \Delta = +1$ and z lies in $\mathrm{Spin}^1(V_0)$, then $\mathrm{Spin}^1(V_0)$ maps onto the identity component $PO^1(V_0) = O^1(V_0)/\{\pm 1\}$ of $PO(V_0)$ with kernel $\{\pm 1, \pm z\}$. If the number of such e_i is odd, so that neither z lies in $\mathrm{Spin}^1(V_0)$ nor -1 in $O^1(V_0)$, then $PO^1(V_0)$ and $O^1(V_0)$ are isormorphic and $\mathrm{Spin}^1(V_0)$ maps onto $PO^1(V_0)$ with kernel $\{\pm 1\}$. If W_0 is a nonsingular subspace of V_0 of even dimension, the inclusion of $\mathrm{Spin}(W_0)$ in $\mathrm{Spin}(V_0)$ corresponds to the inclusion of $O(W_0)$ in $PO(V_0)$. The same assertions hold for the other two spin groups.

Let V be a complex vector space with a complex conjugation $x \to {}^c x$ as in Section 3.3. The conjugation $u \to {}^c u$ on linear transformations of V induces one on linear transformations modulo complex scalars by ${}^c(u \bmod \mathbb{C}^* \cdot 1) = {}^c u \bmod \mathbb{C}^* \cdot 1$. $u \bmod \mathbb{C}^* \cdot 1$ is real if ${}^c u = \lambda u$, $\lambda \neq 0$ complex. This conjugation sends $PO(V) = O(V) \cdot \mathbb{C}^*/\mathbb{C}^* \cdot 1$ to itself; $PO(V_0) = O(V_0) \cdot \{\mu 1\}/\{\mu 1\}$, $|\mu| = 1$, in $PO(V)$ is the subgroup of real elements. The preimages in $D(V)$, under

$g \to p(g) \bmod\{\pm 1\}$, of the various projective orthogonal groups of V_0 are the same as those of the corresponding orthogonal groups.

3.5. For later use, we employ the considerations of Sections 3.1–3.3 to determine the spin groups of Minkowski space.

For any 2×2 matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathbb{C}^{(2,2)}$, set ${}^sM = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$. ${}^sM = {}^tE^{-1}\,{}^tM{}^tE = E{}^tME^{-1}$ is the transpose of M with respect to any alternating form ${}^txEy$ on $\mathbb{C}^2$: ${}^t({}^sMx)Ey = {}^txE(My)$; we will always use $E = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Observe that: ${}^sMM = M{}^sM = (\det M)1_2$, so ${}^sM = M^{-1}$ if $\det M = 1$; ${}^s({}^sM) = M$ and ${}^s(NM) = {}^sM{}^sN$; and ${}^{st}M = {}^{ts}M = \left(\begin{smallmatrix} d & -c \\ -b & a \end{smallmatrix}\right)$.

In the following sections, W denotes the complex vector space of dimension 4 consisting of 4×4 matrices $x = \left(\begin{smallmatrix} 0 & X \\ {}^sX & 0 \end{smallmatrix}\right)$ in $\mathbb{C}^{(4,4)}$, X in $\mathbb{C}^{(2,2)}$, having the quadratic form $(x|x) = -\det X$.

3.6. Since $x^2 = (\det X)1_4 = -(x|x)1_4$, the inclusion of W in $\mathbb{C}^{(4,4)}$ extends to an algebra homomorphism, necessarily injective, of $C(W)$ into $\mathbb{C}^{(4,4)}$. This homomorphism is onto since $C(W)$ and $\mathbb{C}^{(4,4)}$ have dimensions $2^4 = 16$ and $4^2 = 16$, respectively. We identify $C(W) = \mathbb{C}^{(4,4)}$.

Let e_0, e_1, e_2, e_3 be the basis of W given by $\left(\begin{smallmatrix} 0 & X \\ {}^sX & 0 \end{smallmatrix}\right)$ with $X = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, respectively. This basis is orthonormal, $(e_i|e_j) = \eta_{ij}$, where $\eta = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1_3 \end{smallmatrix}\right)$, $\Delta = -1$, and $z = e_0e_1e_2e_3 = \left(\begin{smallmatrix} i1_2 & 0 \\ 0 & -i1_2 \end{smallmatrix}\right)$. For x in W, $zxz^{-1} = -x$, so ${}^Ia = zaz^{-1}$, a in $\mathbb{C}^{(4,4)}$, is the involution $a \to {}^Ia$ of $C(W)$. For $a = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, $A, \ldots, D$ in $\mathbb{C}^{(2,2)}$, we have ${}^Ia = \left(\begin{smallmatrix} A & -B \\ -C & D \end{smallmatrix}\right)$.

To determine $D(W)$, it suffices to determine $D^+(W)$, since 1_4 and e_0 are representatives of its cosets. For g in $C(W)$ even and invertible, $g = \left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right)$, $\det A \neq 0, \det D \neq 0$. For $x = \left(\begin{smallmatrix} 0 & X \\ {}^sX & 0 \end{smallmatrix}\right)$ in W, we have $gx{}^Ig^{-1} = \left(\begin{smallmatrix} 0 & AXD^{-1} \\ D{}^sXA^{-1} & 0 \end{smallmatrix}\right)$. This lies in W when ${}^s(AXD^{-1}) = D{}^sXA^{-1}$ or ${}^sX{}^sAA = {}^sDD{}^sX$; that is, $\det A = \det D$. Hence,

$$D^+(W) = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \middle| \begin{array}{l} A, D \text{ in } \mathbb{C}^{(2,2)} \\ \det A = \det D \neq 0 \end{array} \right\}$$

and, consequently,

$$D(W) = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \text{ or } \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \middle| \begin{array}{l} \det A = \det D \neq 0 \\ \det B = \det C \neq 0 \end{array} \right\}$$

3.7. In terms of the basis e_0, e_1, e_2, e_3 of 3.6, set $k = e_0e_1e_3 = \left(\begin{smallmatrix} 0 & E \\ E & 0 \end{smallmatrix}\right)$. Then $k\bar{e}_ik^{-1} = e_i$, so ${}^cx = k\bar{x}k^{-1}$ is a complex conjugation of W having as real subspace W_0 the real subspace of W spanned by e_0, e_1, e_2, e_3. W_0 consists of $x = e_0x^0 + e_1x^1 + e_2x^2 + e_3x^3 = \left(\begin{smallmatrix} 0 & X \\ {}^sX & 0 \end{smallmatrix}\right)$ with

$$X = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}, \qquad x^0, \ldots, x^3 \text{ real}$$

Note that $(x|x) = -\det X = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2$, so W_0 is Minkowski space. The conjugation on W induces the conjugation ${}^c a = k\bar{a}k^{-1}$ on $C(W)$. For $a = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in $C(W)$, $A, \ldots, D$ in $\mathbb{C}^{(2,2)}$, we have

$${}^c a = k\bar{a}k^{-1} = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} \overline{\begin{pmatrix} A & B \\ C & D \end{pmatrix}} \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}^{-1} = \begin{pmatrix} E\bar{D}E^{-1} & E\bar{C}E^{-1} \\ E\bar{B}E^{-1} & E\bar{A}E^{-1} \end{pmatrix} = \begin{pmatrix} {}^{st}\bar{D} & {}^{st}\bar{C} \\ {}^{st}\bar{B} & {}^{st}\bar{A} \end{pmatrix}$$

Especially, for $x = \begin{pmatrix} 0 & X \\ {}^sX & 0 \end{pmatrix}$ in W, ${}^c x = \begin{pmatrix} 0 & {}^t\bar{X} \\ {}^{st}\bar{X} & 0 \end{pmatrix}$; x is real when ${}^t\bar{X} = X$. $C(W_0)$ is the real 16-dimensional subalgebra

$$C(W_0) = \left\{ \begin{pmatrix} A & B \\ {}^{st}\bar{B} & {}^{st}\bar{A} \end{pmatrix} \middle| \; A, B \text{ in } \mathbb{C}^{(2,2)} \right\}$$

of $\mathbb{C}^{(4,4)}$. Since $\det {}^{st}\bar{A} = \overline{\det A}$, we have

$$D(W_0) = \left\{ \begin{pmatrix} A & 0 \\ 0 & {}^{st}\bar{A} \end{pmatrix} \text{ or } \begin{pmatrix} 0 & B \\ {}^{st}\bar{B} & 0 \end{pmatrix} \middle| \; \begin{matrix} \det A \neq 0 \text{ and real} \\ \det B \neq 0 \text{ and real} \end{matrix} \right\}$$

and

$$D^+(W_0) = \left\{ \begin{pmatrix} A & 0 \\ 0 & {}^{st}\bar{A} \end{pmatrix} \middle| \; \begin{matrix} A \text{ in } \mathbb{C}^{(2,2)} \\ \det A \neq 0 \text{ and real} \end{matrix} \right\}$$

The latter group has two components where $\det A$ is positive or negative.

3.8. For $a = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in $C(W)$, $A, \ldots, D$ in $\mathbb{C}^{(2,2)}$, we have

$$\begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} {}^t a \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}^{-1} = \begin{pmatrix} E{}^tAE^{-1} & E{}^tCE^{-1} \\ E{}^tBE^{-1} & E{}^tDE^{-1} \end{pmatrix} = \begin{pmatrix} {}^sA & {}^sC \\ {}^sB & {}^sD \end{pmatrix}$$

$a \to \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} {}^t a \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}^{-1}$ is an anti-involution of $C(W)$ which fixes each x in W; it is the anti-involution $a \to {}^J a$: ${}^J a = \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} {}^t a \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}^{-1}$. Note that ${}^{IJ}a = \begin{pmatrix} {}^sA & -{}^sC \\ -{}^sB & {}^sD \end{pmatrix}$.

For $g = \begin{pmatrix} A & 0 \\ 0 & {}^{st}\bar{A} \end{pmatrix}$ and $\begin{pmatrix} 0 & B \\ {}^{st}\bar{B} & 0 \end{pmatrix}$ in $D(W_0)$, we have $Ng = \det A$ and $-\det B$, respectively. Hence

$$\text{Spin}\,(W_0) = \left\{ \begin{pmatrix} A & 0 \\ 0 & {}^{st}\bar{A} \end{pmatrix} \text{ or } \begin{pmatrix} 0 & B \\ {}^{st}\bar{B} & 0 \end{pmatrix} \middle| \; \begin{matrix} \det A = \pm 1 \\ \det B = \pm 1 \end{matrix} \right\}$$

$$\text{Spin}^+(W_0) = \left\{ \begin{pmatrix} A & 0 \\ 0 & {}^{st}\bar{A} \end{pmatrix} \middle| \; \begin{matrix} A \text{ in } \mathbb{C}^{(2,2)} \\ \det A = \pm 1 \end{matrix} \right\}$$

$$\text{Spin}^1(W_0) = \left\{ \begin{pmatrix} A & 0 \\ 0 & {}^{st}\bar{A} \end{pmatrix} \middle| \; \begin{matrix} A \text{ in } \mathbb{C}^{(2,2)} \\ \det A = +1 \end{matrix} \right\}$$

When $\det A = +1$, ${}^{st}\bar{A} = {}^t\bar{A}^{-1}$, so the last group consists of matrices $\begin{pmatrix} A & 0 \\ 0 & {}^t\bar{A}^{-1} \end{pmatrix}$, $\det A = +1$, and is isomorphic to $SL(2;\mathbb{C})$. $SL(2;\mathbb{C})$ is connected, simply connected, and of real dimension 6. Spin (W_0) is of real dimension 6 and has four components, represented by

$$1_4 = \begin{pmatrix} 1_2 & 0 \\ 0 & 1_2 \end{pmatrix}, \quad \det A = +1; \qquad e_1e_2e_3 = \begin{pmatrix} 0 & -i1_2 \\ i1_2 & 0 \end{pmatrix}, \quad \det B = -1$$

$$e_0 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}, \quad \det B = +1; \qquad z = e_0e_1e_2e_3 = \begin{pmatrix} i1_2 & 0 \\ 0 & -i1_2 \end{pmatrix}, \quad \det A = -1$$

z does not lie in $\mathrm{Spin}^1(W_0)$ since $\Delta = -1$.

3.9. The homomorphism p, $p(g)x = gx^Ig^{-1}$, maps $D(W)$ and $D(W_0)$ onto $O(W)$ and $O(W_0)$ with kernels $\mathbb{C}^* \cdot 1_4$ and $\mathbb{R}^* \cdot 1_4$, respectively. If we refer $p(g)$ to the basis e_0, e_1, e_2, e_3 of 3.6,

$$ge_j^Ig^{-1} = \sum_{i=0}^{3} e_i(p(g))^i{}_j$$

$v = p(g)$ is a complex or real 4×4 matrix, respectively, satisfying ${}^tv\eta v = \eta$. That is, p maps Spin(W_0) onto the full Lorentz group $O(1, 3; \mathbb{R})$ with kernel $\{\pm 1_4\}$. This latter group has four components, images of the four components of Spin(W_0), according to

$$p(1_4) = 1_4, \qquad p(e_1e_2e_3) = \begin{pmatrix} 1 & 0 \\ 0 & -1_3 \end{pmatrix}$$

$$p(e_0) = \begin{pmatrix} -1 & 0 \\ 0 & 1_3 \end{pmatrix}, \qquad p(z) = -1_4$$

Finally, for $g = \begin{pmatrix} A & 0 \\ 0 & {}^t\bar{A}^{-1} \end{pmatrix}$, $\det A = +1$, in $\mathrm{Spin}^1(W_0)$, and $x = \begin{pmatrix} 0 & X \\ {}^s X & 0 \end{pmatrix}$, ${}^t\bar{X} = X$, in W_0, the linear transformation $x' = gx^Ig^{-1}$ is the familiar spinor form $X' = AX{}^t\bar{A}$ of a proper Lorentz transformation.

4. *Skew-Symmetric Matrices*

4.1. For a skew 4×4 matrix $X = (X^{ij})$ in $\mathbb{C}^{(4,4)}$, representing an alternating form on row vectors, define the associated matrix ${}^aX$ to be the skew 4×4 matrix, representing an alternating form on column vectors, with entries

$$\frac{1}{2} \sum_{\alpha,\beta=1}^{4} \epsilon_{ij\alpha\beta} X^{\alpha\beta}$$

where $\epsilon_{ij\alpha\beta}$ is 1 or -1 if $ij\alpha\beta$ is an even or odd permutation of 1234 and 0 otherwise. If

$$X = \left(\begin{array}{cc|cc} 0 & X^{12} & X^{13} & X^{14} \\ -X^{12} & 0 & X^{23} & -X^{42} \\ \hline -X^{13} & -X^{23} & 0 & X^{34} \\ -X^{14} & X^{42} & -X^{34} & 0 \end{array}\right)$$

then

$$ {}^aX = \left(\begin{array}{cc|cc} 0 & X^{34} & X^{42} & X^{23} \\ -X^{34} & 0 & X^{14} & -X^{13} \\ \hline -X^{42} & -X^{14} & 0 & X^{12} \\ -X^{23} & X^{13} & -X^{12} & 0 \end{array}\right) $$

Similarly, for skew $U = (U_{ij})$, define ${}^aU$ to be the skew matrix with entries

$$ \frac{1}{2} \sum_{\alpha,\beta=1}^{4} \epsilon^{ij\alpha\beta} U_{\alpha\beta} $$

where $\epsilon^{ij\alpha\beta}$ is defined like $\epsilon_{ij\alpha\beta}$. The display of ${}^aU$ is the same as that of ${}^aX$. Note that ${}^a({}^aX) = X$, ${}^a({}^aU) = U$, and that $\det {}^aX = \det X$. In the notation of 3.5, if

$$ X = \begin{pmatrix} aE & B \\ -{}^tB & dE \end{pmatrix} $$

then

$$ {}^aX = \begin{pmatrix} dE & -{}^{st}B \\ {}^sB & aE \end{pmatrix} $$

a and d complex, B in $\mathbb{C}^{(2,2)}$.

For a skew 4 x 4 matrix X, define the Pfaffian of X by

$$ \mathrm{pf} X = \tfrac{1}{8}\Sigma \epsilon_{ij\alpha\beta} X^{ij} X^{\alpha\beta} = X^{12}X^{34} + X^{13}X^{42} + X^{14}X^{23} $$

likewise define $\mathrm{pf}\,U$. Note that $\mathrm{pf}\,{}^aX = \mathrm{pf}\,X$ and ${}^aXX = X{}^aX = -(\mathrm{pf}X)1_4$. Consequently, $(\det X)^2 = \det({}^aXX) = (\mathrm{pf}X)^4$, so $\det X = (\mathrm{pf}X)^2$. If

$$ X = \begin{pmatrix} aE & B \\ -{}^tB & dE \end{pmatrix} $$

then $\mathrm{pf}X = ad - \det B$.

The key formula for the development to follow is, for $X = (X^{ij})$ skew and $A = (A^i{}_j)$ arbitrary in $\mathbb{C}^{(4,4)}$,

$$ {}^a({}^tA{}^a(AX{}^tA)A) = (\det A)X $$

This is immediately proved using $\epsilon_{abrs}A^a{}_mA^b{}_nA^r{}_\alpha A^s{}_\beta = (\det A)\epsilon_{mn\alpha\beta}$, summation signs omitted, and developing $\frac{1}{2}\epsilon^{ijmn}A^a{}_m(\frac{1}{2}\epsilon_{abrs}A^r{}_\alpha X^{\alpha\beta}A^s{}_\beta)A^b{}_n$ to obtain $(\det A)X^{ij}$. For A invertible, $\det A \neq 0$, the formula gives ${}^a(AX{}^tA) = (\det A){}^tA^{-1}{}^aXA^{-1}$.

As a consequence of this last formula, $\mathrm{pf}(AX{}^tA) = (\det A)\,(\mathrm{pf}\,X)$. We may assume $\det A \neq 0$, since otherwise both sides are zero. Then $-\mathrm{pf}(AX{}^tA)1_4 = {}^a(AX{}^tA)(AX{}^tA) = (\det A)({}^tA^{-1}\,{}^aXA^{-1})(AX{}^tA) = (\det A){}^tA^{-1}(-(\mathrm{pf}X)1_4){}^tA = -(\det A)(\mathrm{pf}X)1_4$.

In the following sections V denotes the complex vector space of dimension 6 consisting of 8 x 8 matrices $x = \begin{pmatrix} 0 & X \\ -{}^aX & 0 \end{pmatrix}$ in $\mathbb{C}^{(8,8)}$, where X in $\mathbb{C}^{(4,4)}$ is skew, and having the quadratic form $(x|x) = -\mathrm{pf}X$.

4.2. Since $x^2 = (\mathrm{pf}X)1_8 = -(x|x)1_8$, the same argument as in 3.6 shows $C(V)$ is a subalgebra of $\mathbb{C}^{(8,8)}$. $C(V)$ and $\mathbb{C}^{(8,8)}$ have dimensions $2^6 = 64$ and $8^2 = 64$, respectively, so we identify $C(V) = \mathbb{C}^{(8,8)}$.

Set $z = \begin{pmatrix} -i1_4 & 0 \\ 0 & i1_4 \end{pmatrix}$ in $C(V)$. For x in V, $zxz^{-1} = -x$, so ${}^I a = zaz^{-1}$, a in $\mathbb{C}^{(8,8)}$, is the involution $a \to {}^I a$ of $C(V)$. For $a = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, $A, \ldots, D$ in $\mathbb{C}^{(4,4)}$, we have ${}^I a = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix}$.

Preliminary to determining $D(V)$, observe that if M is a matrix such that MX is skew for all skew X, then M is a scalar matrix. Consequently, if AXD^{-1} is skew for all skew X, so is ${}^tDAX = {}^tD(AXD^{-1})D$, and then ${}^tDA$ is a scalar matrix.

Let g in $C(V)$ be even and invertible, $g = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$, $\det A \neq 0, \det D \neq 0$. For $x = \begin{pmatrix} 0 & X \\ -{}^aX & 0 \end{pmatrix}$ in V, we have $gx^Ig^{-1} = \begin{pmatrix} 0 & AXD^{-1} \\ -D{}^aXA^{-1} & 0 \end{pmatrix}$. For this to lie in V for all x, AXD^{-1} must be skew for all skew X and ${}^a(AXD^{-1}) = D{}^aXA^{-1}$. From the remark above, ${}^tDA = \rho 1_4$, $\rho \neq 0$ complex, and then ${}^a(AX(\rho{}^tA^{-1})^{-1}) = (\rho{}^tA^{-1}){}^aXA^{-1}$ or ${}^a(AX{}^tA) = \rho^2\,{}^tA^{-1}\,{}^aXA^{-1}$. Compare this with the key formula of 4.1 to conclude $\det A = \rho^2$, and then $\det D = \rho^4/\det A = \rho^2$. The same argument applied to $g = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$ odd in $C(V)$ yields the conditions ${}^tCB = \rho 1_4$, $\rho \neq 0$ complex, $\det B = \det C = \rho^2$. Hence

$$D(V) = \left\{ \begin{matrix} \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \\ \text{or} \\ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \end{matrix} \;\middle|\; \begin{matrix} A, B, C, D \text{ in } \mathbb{C}^{(4,4)} \\ {}^tDA = \rho 1_4, \rho \neq 0 \text{ complex} \\ \det A = \det D = \rho^2 \\ {}^tCB = \rho 1_4, \rho \neq 0 \text{ complex} \\ \det B = \det C = \rho^2 \end{matrix} \right\}$$

The matrices of the first type constitute $D^+(V)$.

4.3. The homomorphism p, $p(g)x = gx^Ig^{-1}$, maps $D(V)$ and $D^+(V)$ onto $O(V)$ and $O^+(V)$, respectively, with kernel $\mathbb{C}^* \cdot 1_8$. When x in V is referred to the coordinates X^{ij} of X, the quadratic form $(x|x) = -\mathrm{pf}X = -(X^{12}X^{34} + X^{13}X^{42} + X^{14}X^{23})$ has matrix $\begin{pmatrix} 0 & -\frac{1}{2}1_3 \\ -\frac{1}{2}1_3 & 0 \end{pmatrix}$ with signature $(3, 3)$, so $O(V)$ and $O^+(V)$ are isomorphic to $O(3, 3; \mathbb{C})$ and $O^+(3, 3; \mathbb{C})$; cf. Section 2.8. Hence

$$D(V)/\mathbb{C}^* \cdot 1_8 \simeq O(3, 3; \mathbb{C}) \text{ and } D^+(V)/\mathbb{C}^* \cdot 1_8 \simeq O^+(3, 3; \mathbb{C})$$

If we factor out additionally z and its image $p(z) = -1_6$, we obtain

$$D(V)/\mathbb{C}^* \cdot \{1_8, z\} \simeq O(3, 3; \mathbb{C})/\{\pm 1_6\} = PO(3, 3; \mathbb{C})$$

and

$$D^+(V)/\mathbb{C}^* \cdot \{1_8, z\} \simeq PO^+(3, 3; \mathbb{C})$$

$D(V)$ contains the subgroup

$$G = \left\{ \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} \text{ or } \begin{pmatrix} 0 & B \\ -{}^tB^{-1} & 0 \end{pmatrix} \;\middle|\; \begin{matrix} \det A = +1 \\ \det B = +1 \end{matrix} \right\}$$

with identity component $G^+ = D^+(V) \cap G$ isormorphic to $SL(4, \mathbb{C})$. p maps G and G^+ onto $O(V)$ and $O^+(V)$ with kernel $\{\pm 1_8\}$. Hence the isomorphisms

$$G/\{\pm 1_8\} \simeq O(3, 3; \mathbb{C}), \qquad G/\{\pm 1_8, \pm z\} \simeq PO(3, 3; \mathbb{C})$$

and

$$SL(4; \mathbb{C})/\{\pm 1_4\} \simeq G^+/\{\pm 1_8\} \simeq O^+(3, 3; \mathbb{C})$$

$$PSL(4; \mathbb{C}) = SL(4; \mathbb{C})/\{\pm 1_4, \pm i 1_4\} \simeq G^+/\{\pm 1_8, \pm z\} \simeq PO^+(3, 3; \mathbb{C})$$

These isomorphisms correspond to the isomorphism $A_3 \simeq D_3$ of simple complex Lie algebras.

4.4. For $a = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ in $C(V)$, $A, \ldots, D$ in $\mathbb{C}^{(4,4)}$, we have

$$\begin{pmatrix} 0 & 1_4 \\ -1_4 & 0 \end{pmatrix} {}^t a \begin{pmatrix} 0 & 1_4 \\ -1_4 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} {}^tD & -{}^tB \\ -{}^tC & {}^tA \end{pmatrix}$$

$a \to \left(\begin{smallmatrix} 0 & 1_4 \\ -1_4 & 0 \end{smallmatrix}\right) {}^t a \left(\begin{smallmatrix} 0 & 1_4 \\ -1_4 & 0 \end{smallmatrix}\right)^{-1}$ is an anti-involution of $C(V)$ which fixes each x in V; it is the anti-involution $a \to {}^J a$: ${}^J a = \left(\begin{smallmatrix} 0 & 1_4 \\ -1_4 & 0 \end{smallmatrix}\right) {}^t a \left(\begin{smallmatrix} 0 & 1_4 \\ -1_4 & 0 \end{smallmatrix}\right)^{-1}$. Note that ${}^{IJ}a = \left(\begin{smallmatrix} {}^tD & {}^tB \\ {}^tC & {}^tA \end{smallmatrix}\right)$.

5. *Line Geometry*

5.1. A skew 4 x 4 matrix $X = (X^{ij})$ in $\mathbb{C}^{(4,4)}$, viewed as a bivector, is decomposable, that is, of the form $X = x^t y - y^t x$ or $X^{ij} = x^i y^j - y^i x^j$ with x and y column vectors of $\mathbb{C}^4$, exactly when $\operatorname{pf} X = 0$. Up to a nonzero complex scalar factor, X then depends only on the 2-space of $\mathbb{C}^4$ spanned by x and y. X is the matrix $P = x^t y - y^t x$ of Plücker coordinates $p^{ij} = x^i y^j - y^i x^j$ of this 2-space or of the corresponding line joining the points x and y in complex projective space P^3. Dually, a 2-space of $\mathbb{C}^4$ which is the intersection of two 3-spaces $ux' = 0$ and $vx' = 0$, u and v row covectors of $\mathbb{C}^4$, has dual Plücker coordinates $Q = {}^t uv - {}^t vu$ or $q_{ij} = u_i v_j - v_i u_j$; these are also the dual Plücker coordinates of the corresponding line in P^3 given as the intersection of two planes.

With Q as above, $Qx' = {}^t u(vx') - {}^t v(ux')$, so $Qx' = 0$ for any x' in the intersection of $ux' = 0$ and $vx' = 0$; and conversely since u and v are independent. $Qx' = 0$ is the equation of the 2-space in $\mathbb{C}^4$ or the line in P^3 having dual Plücker coordinates Q. Dually, $u'P = 0$ gives the 3-spaces of $\mathbb{C}^4$ or the planes of P^3 containing the 2-space of $\mathbb{C}^4$ or line of P^3 having Plücker coordinates P.

If $P = x^t y - y^t x$ and $Q = {}^t uv - {}^t vu$ describe the same 2-space of $\mathbb{C}^4$ or the same line of P^3, then $Q = \lambda {}^a P$ with $\lambda \neq 0$ complex. For, ${}^a PP = -(\operatorname{pf} P) 1_4 = 0$ and $QP = {}^t u(vx)^t y - \cdots + {}^t v(uy)^t x = 0$; and the skew matrices X satisfying $XP = 0$ constitute a one-dimensional space, as may be verified by choosing a basis so that

$$P = \left(\begin{array}{c|c} \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & 0 \\ \hline 0 & 0 \end{array}\right)$$

5.2. A collineation of complex projective space P^3 is described by a linear transformation $x' = Ax$ of column vectors x of $\mathbb{C}^4$ representing points and simultaneously a linear transformation $u' = u^tD$ of row covectors u of $\mathbb{C}^4$ representing planes. In order that incidence be preserved, one requires $u'x' = u^tDAx$ to be a nonzero scalar multiple of ux, or ${}^tDA = \rho 1_4$, $\rho \neq 0$ complex. Let $\mu^2 = \rho^2/\det A = \det D/\rho^2$; then ${}^tD(\mu A) = (\mu\rho)1_4$ and $\det(\mu A) = \det D = (\mu\rho)^2$. Multiplying A or D by a nonzero scalar does not change the collineation, so we may replace A by μA and thus arrange A and D to satisfy ${}^tDA = \rho 1_4$, $\det A = \det D = \rho^2$, $\rho \neq 0$ complex.

A correlation of P^3, which sends points to planes and planes to points and preserves incidence, is similarly described by $u' = {}^tx^tC$ and $x' = B^tu$, where we may arrange ${}^tCB = \rho 1_4$, $\det B = \det C = \rho^2$, $\rho \neq 0$ complex.

All collineations and correlations of P^3 constitute a group, and, if we describe the collineations and correlations above by the matrices $\left(\begin{smallmatrix} A & 0 \\ 0 & D\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & B \\ C & 0\end{smallmatrix}\right)$, respectively, a succession of transformations is described by the product of the corresponding matrices. We thus have a homomorphism from the group $D(V)$ of Section 4.2 onto the group of collineations and correlations of P^3. An element of the kernel of this homomorphism is of the form $\left(\begin{smallmatrix} A & 0 \\ 0 & D\end{smallmatrix}\right)$, with $A = \alpha 1_4$, $D = \delta 1_4$, α and δ scalars, and the conditions ${}^tDA = \rho\, 1_4$, $\det A = \det D = \rho^2$ yield $\alpha^4 = \delta^4 = (\alpha\delta)^2$ or $\delta = \pm\alpha$; the kernel is $\mathbb{C}^* \cdot \{1_8, z\}$, $z = \left(\begin{smallmatrix} -i1_4 & 0 \\ 0 & i1_4\end{smallmatrix}\right)$. Hence we have the isomorphism

$$D(V)/\mathbb{C}^* \cdot \{1_8, z\} \simeq \begin{pmatrix}\text{collineations and} \\ \text{correlations of } P^3\end{pmatrix}$$

5.3. The linear transformations describing a collineation, $x' = Ax$ and $u' = u^tD$, where ${}^tDA = \rho 1_4$ and $\det A = \det D = \rho^2$, induce linear transformations on skew matrices representing bivectors $X = (X^{ij})$ and bicovectors $U = (U_{ij})$ by $X' = AX^tA$ and $U' = DU^tD$; this is consistent with $X = x^ty - y^tx$ and $U = {}^tuv - {}^tvu$ when X or U is decomposable. Since ${}^tDA = \rho 1_4$, these are $X' = \rho AXD^{-1}$ and $U' = \rho DUA^{-1}$. When $-{}^aX = U$, we have

$$-{}^aX' = -{}^a(AX^tA) = -(\det A){}^tA^{-1a}XA^{-1} = \rho^2\rho^{-1}D(-{}^aX)\rho^{-1t}D = DU^tD = U'$$

by the key formula of Section 4.1. In this case we may write the linear transformations as

$$\begin{pmatrix} 0 & X' \\ -{}^aX' & 0 \end{pmatrix} = \rho \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} 0 & X \\ -{}^aX & 0 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}^{-1}$$

Similarly, the linear transformations describing a correlation, $u' = {}^tx^tC$ and $x' = B^tu$, where ${}^tCB = \rho 1_4$ and $\det B = \det C = \rho^2$, induce linear transformations between skew matrices representing bivectors and bicovectors by $U' = CX^tC$ and $X' = BU^tB$. Since ${}^tCB = \rho 1_4$, these are $U' = \rho CXB^{-1}$ and $X' = \rho BUC^{-1}$. Again, when $-{}^aX = U$, we have $-{}^aX' = U'$. In this case we may write the linear transformations as

$$\begin{pmatrix} 0 & X' \\ -{}^aX' & 0 \end{pmatrix} = \rho \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \begin{pmatrix} 0 & X \\ -{}^aX & 0 \end{pmatrix} \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}^{-1}$$

5.4. Let V be the complex vector space of dimension 6 of Section 4.2 ff., so x in V is $x = \begin{pmatrix} 0 & X \\ -{}^aX & 0 \end{pmatrix}$, X in $\mathbb{C}^{(4,4)}$ skew, and $(x|x) = -\mathrm{pf}X$. Denote by Ω the Plücker quadric $(x|x) = 0$, of complex dimension 4, in the complex projective space P^5 based on V. A point x of Ω represents a line of P^3, X and $-{}^aX$ being the matrices of its Plücker coordinates and dual coordinates, respectively, as in 5.1.

The collineations of P^5 which send Ω to itself are described by linear transformations $x' = \lambda ux$, u in $O(V)$, $\lambda \neq 0$ complex. By 4.3 u is $ux = gx^Ig^{-1}$, g in $D(V)$, and by Section 5.3, $x' = \lambda gx^Ig^{-1} = \pm\lambda gxg^{-1}$ determines a collination or correlation of P^3. Consequently, the isomorphism of Sections 3.4 and 4.3, between $D(V)/\mathbb{C}^* \cdot \{1_8, z\}$ and $PO(V) \simeq PO(3, 3; \mathbb{C})$, is between the group of collineations and correlations of P^3 and the group of collineations of P^5 that preserve Ω.

Geometrically, a point x of Ω represents a line of P^3 described as the line joining two points or as the line of intersection of two planes. A collineation or correlation of P^3 sends this line into another line similarly described. It is represented on Ω by a point that is the image of x under the corresponding collineation of P^5 which preserves Ω.

5.5. Let H be Hermitean, ${}^t\bar{H} = H$, and of determinant $\det H = +1$. The antilinear transformations $u' = {}^t\bar{x}(iH)$ and $x' = (iH)^{-1}\,{}^t\bar{u}$ describe an anticorrelation of P^3 that is involutive, that is, an antipolarity, since $(iH)^{-1}\,{}^t(\overline{{}^t\bar{x}(iH)}) = -H^{-1}\,{}^t\bar{H}x = -x$ and ${}^t(\overline{(iH)^{-1}\,{}^t\bar{u}})(iH) = -u\,{}^t\bar{H}^{-1}H = -u$ are scalar multiples of x and u.

The antilinear transformations above induce antilinear transformations, again involutive, between skew matrices representing bivectors and bicovectors by $U' = {}^t(iH)\bar{X}(iH) = -{}^tH\bar{X}H$ and $X' = (iH)^{-1}\bar{U}^t(iH)^{-1} = -H^{-1}\bar{U}^tH^{-1}$. When $-{}^aX = U$, we have

$$-{}^aX' = -{}^a(-H^{-1}\bar{U}^tH^{-1}) = (\det H^{-1})^tH^a\bar{U}H = -{}^tH\bar{X}H = U'$$

by the key formula of 4.1. In this case we may write the antilinear transformations as

$$\begin{pmatrix} 0 & X' \\ -{}^aX' & 0 \end{pmatrix} = \begin{pmatrix} 0 & (iH)^{-1} \\ {}^t(iH) & 0 \end{pmatrix} \overline{\begin{pmatrix} 0 & X \\ -{}^aX & 0 \end{pmatrix}} \begin{pmatrix} 0 & (iH)^{-1} \\ {}^t(iH) & 0 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 0 & -H^{-1} \\ {}^tH & 0 \end{pmatrix} \overline{\begin{pmatrix} 0 & X \\ -{}^aX & 0 \end{pmatrix}} \begin{pmatrix} 0 & -H^{-1} \\ {}^tH & 0 \end{pmatrix}^{-1}$$

5.6. For the collineation of P^3 described by the linear transformations $x' = Ax$ and $u' = u^tD$, ${}^tDA = \rho 1_4$, $\det A = \det D = \rho^2$, to commute with the antipolarity of 5.5, it must be that $(iH)^{-1}\,{}^t(\overline{u^tD}) = \lambda A((iH)^{-1}\,{}^t\bar{u})$ or $H^{-1}\bar{D}H = \lambda A$ with $\lambda \neq 0$ complex. Necessarily $|\lambda| = 1$, since $\lambda^4 \det A = \det(\lambda A) = \det(H^{-1}\bar{D}H) = \overline{\det D} = \overline{\det A}$. From $H^{-1}\bar{D}H = \lambda A$ we have $A = \lambda^{-1}H^{-1}\bar{D}H$ and

${}^t H\bar{A}^t H^{-1} = \bar{\lambda}^{-1}\,{}^t H\bar{H}^{-1} D\bar{H}^t H^{-1} = \lambda D$; this gives ${}^t\bar{A}H = \lambda H^t D$ and ${}^t(\overline{Ax})(iH) = \lambda({}^t\bar{x}(iH))^t D$. The condition that the collineation commute with the antipolarity may be written

$$\begin{pmatrix} 0 & -H^{-1} \\ {}^tH & 0 \end{pmatrix} \overline{\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}} \begin{pmatrix} 0 & -H^{-1} \\ {}^tH & 0 \end{pmatrix}^{-1} = \lambda \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$$

where the complex scalar $\lambda \neq 0$ depends on A and D. For such a collineation, set $\lambda = \mu^2$, $|\mu| = 1$. Then ${}^t(\mu D)(\mu A) = (\mu^2\rho)1_4$, $\det(\mu A) = \det(\mu D) = (\mu^2\rho)^2$ and

$$\begin{pmatrix} 0 & -H^{-1} \\ {}^tH & 0 \end{pmatrix} \overline{\begin{pmatrix} \mu A & 0 \\ 0 & \mu D \end{pmatrix}} \begin{pmatrix} 0 & -H^{-1} \\ {}^tH & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \mu A & 0 \\ 0 & \mu D \end{pmatrix}$$

Multiplying A or D by a nonzero scalar does not change the collineation, so we may replace A and D by μA and μD and arrange that the linear transformations describing this collineation satisfy

$$\begin{pmatrix} 0 & -H^{-1} \\ {}^tH & 0 \end{pmatrix} \overline{\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}} \begin{pmatrix} 0 & -H^{-1} \\ {}^tH & 0 \end{pmatrix}^{-1} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$$

Similarly, for the correlation of P^3 described by the linear transformations $u' = {}^t x^t C$ and $x' = B^t u$, ${}^t CB = \rho 1_4$, $\det B = \det C = \rho^2$, to commute with the antipolarity, we obtain the conditions $H^{-1}\bar{C}^t H^{-1} = -\lambda B$ and ${}^t H\bar{B}H = -\lambda C$ with $|\lambda| = 1$. Again we may replace B and C by scalar multiples without changing the correlation and arrange that $\lambda = 1$. The condition on the linear transformations that the correlation commute with the antipolarity may then be written

$$\begin{pmatrix} 0 & -H^{-1} \\ {}^tH & 0 \end{pmatrix} \overline{\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}} \begin{pmatrix} 0 & -H^{-1} \\ {}^tH & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}.$$

5.7. The Hermitean quadric Φ in complex projective space P^3 which is given as a locus of points by ${}^t\bar{x}Hx = 0$ or as an envelope of planes by $\bar{u}^t H^{-1}\,{}^t u = 0$ has real dimension 5 and consists of all points or planes that are incident with their corresponding plane or point under the antipolarity of Section 5.5; that is, $u'x = 0$ for $u' = {}^t\bar{x}(iH)$.

If a collineation or correlation of P^3 commutes with the antipolarity, it necessarily sends Φ to itself. This can also be observed directly from the relations of Section 5.6; for from ${}^t\bar{A}H = H^t D$ and ${}^t DA = \rho\, 1_4$, we obtain ${}^t\bar{A}HA = H^t DA = \rho H$ and ${}^t(\overline{Ax})H(Ax) = \rho\,{}^t\bar{x}Hx$ for collineations, and similarly ${}^t\bar{B}HB = -\rho^t H^{-1}$ and ${}^t(\overline{B^t u})H(B^t u) = -\rho\bar{u}^t H^{-1}\,{}^t u$ for correlations. The converse is also true. Hence the collineations and correlations of P^3 that commute with the antipolarity are exactly those that preserve Φ. The group of these transformations will be described in Section 6.

5.8. The antipolarity of Section 5.5 transforms lines of P^3 into lines of P^3 in the same manner as do correlations in the close of 5.4. If P is the matrix of Plücker coordinates of a line, ${}^t HPH$ is the matrix of dual coordinates of its image by 5.5. Thus a line is fixed under the antipolarity exactly when ${}^t HPH = \lambda^u P$, $\lambda \neq 0$ complex.

The line joining x and y in P^3 lies in Φ, that is, it is a generator of Φ, exactly when ${}^t\bar{x}Hx = 0$, ${}^t\bar{x}Hy = 0$, ${}^t\bar{y}H\bar{x} = 0$, and ${}^t\bar{y}Hy = 0$. Set $P = x{}^ty - y{}^tx$. These are equivalent to $\bar{P}Hx = \bar{x}({}^t\bar{y}Hx) - \bar{y}({}^t\bar{x}Hx) = 0$ and $\bar{P}Hy = 0$ since x and y are independent vectors of $\mathbb{C}^4$. These, in turn, are equivalent to $\bar{P}HP = (\bar{P}Hx){}^ty - (\bar{P}Hy){}^tx = 0$. Hence, a line is a generator exactly when $\bar{P}HP = 0$.

From Section 5.1, the skew matrices X satisfying $XP = 0$ are a one-dimensional space. Since ${}^aPP = -(\mathrm{pf}\,P)1_4 = 0$, ${}^tH\bar{P}H = \lambda{}^aP$ is equivalent to ${}^tH\bar{P}HP = 0$, that is, to $\bar{P}HP = 0$. Consequently the lines of P^3 that are fixed under the antipolarity are exactly the generators of the Hermitean quadric Φ.

6. *The Isomorphism*

Throughout this section, let V, as in Section 4.2 ff., be the complex vector space of dimension 6 of 8×8 matrices $x = \begin{pmatrix} 0 & X \\ -{}^aX & 0 \end{pmatrix}$ in $\mathbb{C}^{(8,8)}$, X in $\mathbb{C}^{(4,4)}$ skew, and having the quadratic form $(x|x) = -\mathrm{pf}\,X$. From Section 4.2, $C(V) = \mathbb{C}^{(8,8)}$ and there also is $D(V)$ described.

6.1. Motivated by the considerations of Sections 5.5 and 5.6, define a complex conjugation $a \to {}^ca$ on $C(V) = \mathbb{C}^{(8,8)}$ by

$$ {}^ca = \begin{pmatrix} 0 & -H^{-1} \\ {}^tH & 0 \end{pmatrix} \bar{a} \begin{pmatrix} 0 & -H^{-1} \\ {}^tH & 0 \end{pmatrix}^{-1} $$

where H is Hermitean ${}^t\bar{H} = H$, so ${}^c({}^ca) = a$. Indeed, this conjugation is the extension to $C(V)$ of the involutive antilinear transformation on V which describes the antipolarity of 5.5 between bivectors and bicovectors. If $a = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, $A, \ldots, D$ in $\mathbb{C}^{(4,4)}$, then

$$ {}^ca = \begin{pmatrix} H^{-1}\bar{D}H & -H^{-1}\bar{C}{}^tH^{-1} \\ -{}^tH\bar{B}H & {}^tH\bar{A}{}^tH^{-1} \end{pmatrix} $$

a is real, ${}^ca = a$, if $C = -{}^tH\bar{B}H$ and $D = {}^tH\bar{A}{}^tH^{-1}$.

Hereafter, H will always denote the Hermitean matrix $H = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}$ of determinant $\det H = +1$.

6.2. From Sections 4.1 and 6.1 an element of V and its conjugate have the form

$$ x = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{0} & aE & B \\ & & -{}^tB & dE \\ \hline -dE & {}^{st}B & \multicolumn{2}{c}{0} \\ -{}^sB & -aE & & \end{array}\right) \quad \text{and} \quad {}^cx = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{0} & \bar{a}E & {}^s\bar{B} \\ & & -{}^{st}\bar{B} & \bar{d}E \\ \hline -\bar{d}E & {}^t\bar{B} & \multicolumn{2}{c}{0} \\ -\bar{B} & -\bar{a}E & & \end{array}\right) $$

x is real, ${}^cx = x$, if a and d are real and ${}^s\bar{B} = B$. Set $B = iX_0E$ with X_0 in $\mathbb{C}^{(2,2)}$. Then ${}^s\bar{B} = (-i)^s\bar{E}{}^s\bar{X}_0 = iE{}^s\bar{X}_0 = i{}^t\bar{X}_0E$, so ${}^s\bar{B} = B$ when ${}^t\bar{X}_0 = X_0$. Set

$$ \begin{array}{l} a = -y + z \\ d = y + z \end{array} \quad \text{and} \quad X_0 = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} $$

Then real x have the form

$$x = \begin{pmatrix} 0 & X \\ -{}^aX & 0 \end{pmatrix}, \qquad X = \begin{pmatrix} (-y+z)E & iX_0E \\ -{}^t(iX_0E) & (y+z)E \end{pmatrix}$$

with $x^0, \ldots, x^3, y, z$ real. These real elements of V constitute the real subspace V_0 of dimension 6.

Note that $\det(iX_0E) = -\det X_0 = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2$. For x in V_0 as above, we have

$$\begin{aligned}(x|x) &= -\mathrm{pf}\,X = -ad + \det B = -(-y+z)(y+z) + \det(iX_0E) \\ &= -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 + y^2 - z^2\end{aligned}$$

The quadratic form $(x|x)$ of V restricted to V_0 has signature (2, 4), that is, $(-++++-)$.

Let $e_0, e_1, e_2, e_3, e_+, e_-$ be the basis of V_0 obtained by taking successively x^0, x^1, x^2, x^3, y, z equal to 1 and the others equal to 0. Thus, for example, e_0 is the 8 x 8 matrix having alternately i and $-i$ along the upper-right to lower-left diagonal and zeros elsewhere. This basis is orthonormal, $(e_i|e_j) = \eta^{\pm}_{ij}$, where

$$\eta^{\pm} = \left(\begin{array}{cc|cc} -1 & 0 & & \\ 0 & 1_3 & \multicolumn{2}{c}{0} \\ \hline & & 1 & 0 \\ \multicolumn{2}{c|}{0} & 0 & -1 \end{array}\right)$$

$\Delta = +1$, and

$$z = e_0e_1e_2e_3e_+e_- = \begin{pmatrix} -i1_4 & 0 \\ 0 & i1_4 \end{pmatrix}$$

6.3. By Section 3.3, $D(V_0)$ consists of the real elements in $D(V)$ of 4.2. For $g = \left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right)$ in $D^+(V)$, ${}^tDA = \rho 1_4$, $\det A = \det D = \rho^2$, $\rho \neq 0$ complex, to be real, ${}^cg = g$, we must have $D = {}^tH\bar{A}^tH^{-1}$ by Section 6.1. From $D = \rho^tA^{-1}$ we obtain ${}^t\bar{A}HA = \rho H$; both ${}^t\bar{A}HA$ and H are Hermitean, so ρ is necessarily real. Set $\rho = \epsilon\lambda^2$, $\epsilon = \pm 1$, $\lambda > 0$ real, then $g = \lambda\left(\begin{smallmatrix} A' & 0 \\ 0 & \epsilon^tA'^{-1} \end{smallmatrix}\right)$, where $A' = \lambda^{-1}A$, $\epsilon^tA'^{-1} = \lambda^{-1}D$, $\det A' = +1$, and ${}^t\bar{A}'HA' = \epsilon H$. Hence

$$D^+(V_0) = \left\{ \lambda\begin{pmatrix} A & 0 \\ 0 & \epsilon^tA^{-1} \end{pmatrix} \;\middle|\; \begin{array}{l} {}^t\bar{A}HA = \epsilon H, \det A = +1 \\ \epsilon = \pm 1, \lambda > 0 \text{ real} \end{array} \right\}$$

In a similar fashion, one obtains for the other coset of $D^+(V_0)$ in $D(V_0)$

$$\left\{ \lambda\begin{pmatrix} 0 & B \\ \epsilon^tB^{-1} & 0 \end{pmatrix} \;\middle|\; \begin{array}{l} {}^t\bar{B}HB = -\epsilon^tH^{-1}, \det B = +1 \\ \epsilon = \pm 1, \lambda > 0 \text{ real} \end{array} \right\}$$

Since ${}^tH^{-1} = H$, or equivalently, $\left(\begin{smallmatrix} 0 & 1_4 \\ -1_4 & 0 \end{smallmatrix}\right)$ is real, the condition on B is the same as that on A.

For $g = \lambda\begin{pmatrix} A & 0 \\ 0 & \epsilon {}^tA^{-1}\end{pmatrix}$ or $\lambda\begin{pmatrix} 0 & B \\ \epsilon {}^tB^{-1} & 0\end{pmatrix}$ in $D(V_0)$, we have from Section 4.4 that $Ng = \lambda^2\epsilon$. Hence the spin groups of V_0 are

$$\mathrm{Spin}(V_0) = \left\{ \begin{array}{c|l} \begin{pmatrix} A & 0 \\ 0 & \epsilon {}^tA^{-1}\end{pmatrix} & \begin{array}{l} {}^t\bar{A}HA = \epsilon H \\ \det A = +1,\ \epsilon = \pm 1 \end{array} \\ & \\ \begin{pmatrix} 0 & B \\ \epsilon {}^tB^{-1} & 0\end{pmatrix} & \begin{array}{l} {}^t\bar{B}HB = -\epsilon {}^tH^{-1} \\ \det B = +1,\ \epsilon = \pm 1 \end{array} \end{array} \right\}$$

$$\mathrm{Spin}^+(V_0) = \left\{ \begin{pmatrix} A & 0 \\ 0 & \epsilon {}^tA^{-1}\end{pmatrix} \;\middle|\; \begin{array}{l} {}^t\bar{A}HA = \epsilon H \\ \det A = +1,\ \epsilon = \pm 1 \end{array} \right\}$$

$$\mathrm{Spin}^1(V_0) = \left\{ \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1}\end{pmatrix} \;\middle|\; \begin{array}{l} {}^t\bar{A}HA = H \\ \det A = +1 \end{array} \right\}$$

Only $\mathrm{Spin}^1(V_0)$ is a subgroup of the group G of Section 4.3.

Denote by $SU(2, 2)$ the group of all matrices A in $\mathbb{C}^{(4,4)}$ for which ${}^t\bar{A}HA = H$ and $\det A = +1$; these are the matrices of the linear transformations of determinant $+1$ which preserve the Hermitean form

$$\begin{aligned} {}^t\bar{x}Hx &= \bar{x}^1x^3 + \bar{x}^2x^4 + \bar{x}^3x^1 + \bar{x}^4x^2 \\ &= \tfrac{1}{2}|x^1 + x^3|^2 - \tfrac{1}{2}|x^1 - x^3|^2 + \tfrac{1}{2}|x^2 + x^4|^2 - \tfrac{1}{2}|x^2 - x^4|^2 \end{aligned}$$

of signature (2, 2). $SU(2, 2)$ is connected and of real dimension 15, but is not simply connected.

$\mathrm{Spin}^1(V_0)$ is isomorphic to $SU(2, 2)$, so $\mathrm{Spin}(V_0)$ and $\mathrm{Spin}^+(V_0)$ have four and two components, respectively. Representatives of these components are

$$1_8 = \left(\begin{array}{cc|cc} 1_2 & 0 & 0 & 0 \\ 0 & 1_2 & 0 & 0 \\ \hline 0 & 0 & 1_2 & 0 \\ 0 & 0 & 0 & 1_2 \end{array}\right),\ \epsilon = +1 \qquad e_1e_2e_3 = \left(\begin{array}{cc|cc} 0 & 0 & 0 & E \\ 0 & 0 & -E & 0 \\ \hline 0 & E & 0 & 0 \\ -E & 0 & 0 & 0 \end{array}\right),\ \epsilon = +1$$

$$e_0 = \left(\begin{array}{cc|cc} 0 & 0 & 0 & iE \\ 0 & 0 & iE & 0 \\ \hline 0 & iE & 0 & 0 \\ iE & 0 & 0 & 0 \end{array}\right),\ \epsilon = -1, \qquad e_0e_1e_2e_3 = \left(\begin{array}{cc|cc} i1_2 & 0 & 0 & 0 \\ 0 & -i1_2 & 0 & 0 \\ \hline 0 & 0 & i1_2 & 0 \\ 0 & 0 & 0 & -i1_2 \end{array}\right),\ \epsilon = -1$$

$z = \begin{pmatrix} -i1_4 & 0 \\ 0 & i1_4 \end{pmatrix}$ lies in $\mathrm{Spin}^1(V_0)$ since $\Delta = +1$.

6.4. The homomorphism p, $p(g)x = gx^Ig^{-1}$, maps $D(V)$ and $D(V_0)$ onto $O(V)$ and $O(V_0)$ with kernels $\mathbb{C}^* \cdot 1_8$ and $\mathbb{R}^* \cdot 1_8$, respectively. If we refer $p(g)$ to the basis $e_0, e_1, e_2, e_3, e_+, e_-$ of Section 6.2, $ge_j^Ig^{-1} = \Sigma_i e_i(p(g))^i{}_j$,

$u = p(g)$ is a complex or real 6×6 matrix, respectively, satisfying ${}^t u \eta^{\pm} u = \eta^{\pm}$, where $\eta^{\pm}$ is as in Section 6.2. Thus p maps $\mathrm{Spin}(V_0)$ onto the real orthogonal group $O(2, 4; \mathbb{R})$, cf. Section 2.8, with kernel $\{\pm 1_8\}$. This latter group has four components, images of the four components of $\mathrm{Spin}(V_0)$ according to

$$p(1_8) = 1_6, \qquad p(e_1e_2e_3) = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -1_3 & \\ \hline \multicolumn{2}{c|}{0} & 1_2 \end{array}\right)$$

$$p(e_0) = \left(\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & 1_3 & \\ \hline \multicolumn{2}{c|}{0} & 1_2 \end{array}\right), \qquad p(e_0e_1e_2e_3) = \begin{pmatrix} -1_4 & 0 \\ 0 & 1_2 \end{pmatrix}$$

These may be obtained by noting that $p(e_i)$ is the reflection in the hyperplane of V_0 orthogonal to e_i and passing through the origin, or by direct calculation using the techniques of Section 6.6 below.

6.5. The homomorphism $g \to p(g) \bmod \{\pm 1_6\}$ maps $\mathrm{Spin}(V_0)$ onto $PO(V_0) = O(V_0)/\{\pm 1_6\}$ with kernel $\{\pm 1_8, \pm z\}$, since z lies in $\mathrm{Spin}(V_0)$ and $p(z) = -1_6$, cf. Section 3.4. Consequently, we have the isomorphism we have been seeking:

$$\mathrm{Spin}(V_0)/\{\pm 1_8, \pm z\} \simeq PO(V_0)$$

This is a real form of the isomorphism corresponding to $A_3 \simeq D_3$ of Section 4.3. Since $\{\pm 1_8, \pm z\}$ and $\{\pm 1_6\}$ lie in the identity components of $\mathrm{Spin}(V_0)$ and $O(V_0)$, respectively, these groups have four components. Since $\mathrm{Spin}^1(V_0)$ is isomorphic to $SU(2, 2)$, as in Section 6.3, the isomorphism restricted to the identity components is

$$PSU(2, 2) = SU(2, 2)/\{\pm 1_4, \pm i1_4\} \simeq O^1(2, 4; \mathbb{R})/\{\pm 1_6\} = PO^1(2, 4; \mathbb{R})$$

Let Φ be the Hermitean quadric ${}^t\bar{x}Hx = \bar{x}^1x^3 + \bar{x}^2x^4 + \bar{x}^3x^1 + \bar{x}^4x^2 = 0$ of real dimension 5 in complex projective space P^3. A point of the Plücker quadric Ω, $(x|x) = 0$, in complex P^5 based on V represents a line of P^3, as in Section 5.4. By Section 5.8, this line is a generator of Φ exactly when it is fixed under the antipolarity of 5.5. From Section 5.5 and 6.1, in terms of x in V which is formed from the matrices of Plücker coordinates and dual coordinates of a line, this antipolarity operating on lines is given by the conjugation $x \to {}^cx$. x in Ω is fixed if ${}^cx$ is a nonzero scalar multiple of x, but, as in Section 3.3, we may arrange that x itself is fixed. Hence, the generators of Φ are represented by the fixed points in Ω of the conjugation $x \to {}^cx$. This set of fixed points is the real quadric Ψ, $(x|x) = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 + y^2 - z^2 = 0$, cf. Section 6.2, in the real projective space ${P_0}^5$ based on V_0.

Collineations and correlations of P^3 correspond to collineations of P^5 which preserve Ω. This yields the isormorphism between $D(V)/\mathbb{C}^* \cdot \{1_8, z\}$ and $PO(V)$ in Section 5.4. Collineations and correlations of P^3 which preserve Φ are those which commute with the antipolarity; as they operate on lines, that is on Ω, they preserve the set of fixed points Ψ of the antipolarity, and hence correspond to collineations of $P_0{}^5$ which preserve Ψ, cf. Section 5.6. This yields the isomorphism between $D(V_0)/\mathbb{R}^* \cdot \{1_8, z\} \simeq \mathrm{Spin}(V_0)/\{\pm 1_8, \pm z\}$ and $PO(V_0)$ above.

Compactified Minkowski space Ψ, cf. Section 2.8, is the space of generators of Φ and the conformal transformations of Ψ are obtained as collineations and correlations permuting these generators: A point of Ψ represents a generator of Φ and the image of this point under a conformal transformation is the point of Ψ that represents the image of the generator under the corresponding collineation or correlation preserving Φ, cf. Section 5.4. This is the geometric description of the isormorphism (Study, 1924, p. 235).

6.6. Rotations, translations, homotheties, and inversions are conformal transformations of Minkowski space and its compactification Ψ and hence are elements of $PO(V_0)$. We can exhibit explicitly elements g of $\mathrm{Spin}(V_0)$ so that $p(g)$ mod$\{\pm 1_6\}$, from the isomorphism of Section 6.5, are these familiar transformations. Cf. Penrose (1967), p. 357 ff.

The technique is the following. For $x = \begin{pmatrix} 0 & X \\ -{}^aX & 0 \end{pmatrix}$ in V_0 and $g = \begin{pmatrix} A & 0 \\ 0 & \epsilon\,{}^tA^{-1} \end{pmatrix}$ in $\mathrm{Spin}^+(V_0)$, ${}^Ig = g$, we have

$$p(g)x = gx^Ig^{-1} = \begin{pmatrix} 0 & \epsilon AX^tA \\ -\epsilon\,{}^tA^{-1}\,{}^aXA^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \epsilon AX^tA \\ -{}^a(\epsilon AX^tA) & 0 \end{pmatrix}$$

so $x' = gx^Ig^{-1}$ is determined by $X' = \epsilon AX^tA$. Similarily, for $g = \begin{pmatrix} 0 & B \\ \epsilon\,{}^tB^{-1} & 0 \end{pmatrix}$, ${}^Ig = -g$, we have $X' = \epsilon B^aX^tB$. These, together with

$$X = \begin{pmatrix} (-y+z)E & iX_0E \\ -{}^t(iX_0E) & (y+z)E \end{pmatrix}, \qquad {}^aX = \begin{pmatrix} (y+z)E & -{}^{st}(iX_0E) \\ {}^s(iX_0E) & (-y+z)E \end{pmatrix}$$

where

$$X_0 = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}$$

cf. Section 6.2, allows $p(g)$ in $O(V_0) = O(2, 4; \mathbb{R})$ to be determined by expressing $x'^0, \ldots, x'^3, y', z'$ linearly in terms of $x^0, \ldots, x^3, y, z$.

Let $g = \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}$, $A = \begin{pmatrix} A_0 & 0 \\ 0 & {}^t\bar{A}_0^{-1} \end{pmatrix}$, with A_0 in $\mathbb{C}^{(2,2)}$, $\det A_0 = +1$. Then ${}^t\bar{A}HA = H$ and $\det A = +1$, so g lies in $\mathrm{Spin}^1(V_0)$. Note that $A_0^{-1} = {}^sA_0$, so $X' = AX^tA$ above becomes

$$X' = \begin{pmatrix} A_0 & 0 \\ 0 & {}^{st}\bar{A}_0 \end{pmatrix} \begin{pmatrix} (-y+z)E & iX_0E \\ -{}^t(iX_0E) & (y+z)E \end{pmatrix} {}^t\begin{pmatrix} A_0 & 0 \\ 0 & {}^{st}\bar{A}_0 \end{pmatrix}$$

$$= \begin{pmatrix} (-y+z)A_0E^tA & iA_0X_0E^s\bar{A}_0 \\ -i^{st}\bar{A}_0{}^t(X_0E)^tA_0 & (y+z)^{st}\bar{A}_0E^s\bar{A}_0 \end{pmatrix}$$

Now $A_0E^tA_0 = A_0{}^sA_0E = (\det A_0)E = E$ and $iA_0X_0E^s\bar{A}_0 = iA_0X_0{}^t\bar{A}_0E$, so we recognize X' as

$$\begin{pmatrix} (-y'+z')E & iX_0'E \\ -{}^t(iX_0'E) & (y'+z')E \end{pmatrix} = \begin{pmatrix} (-y+z)E & iA_0X_0{}^t\bar{A}_0E \\ -{}^t(iA_0X_0{}^t\bar{A}_0E) & (y+z)E \end{pmatrix}$$

or $X_0' = A_0 X_0 {}^t\bar{A}_0, y' = y, z' = z$. Hence $p(g) = \begin{pmatrix} v & 0 \\ 0 & 1_2 \end{pmatrix}$ in $O(V_0)$, where v in $O(1, 3; \mathbb{R})$ is the matrix describing $X_0' = A_0 X_0 {}^t\bar{A}_0$ as in Section 3.9. Thus, g in Spin(V_0) describes a proper rotation or Lorentz transformation of Ψ.

The map

$$\begin{pmatrix} A_0 & 0 \\ 0 & {}^t\bar{A}_0^{-1} \end{pmatrix} \to \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}, \qquad A = \begin{pmatrix} A_0 & 0 \\ 0 & {}^tA_0^{-1} \end{pmatrix}$$

is the inclusion of $\mathrm{Spin}^1(W_0)$ of Section 3.8 in $\mathrm{Spin}^1(V_0)$ which corresponds to the inclusion $v \to \begin{pmatrix} v & 0 \\ 0 & 1_2 \end{pmatrix}$ of $O^1(W_0)$ in $O^1(V_0)$, cf. Section 3.2. The natural extension of this to all of Spin(W_0) will be discussed in Section 6.7.

Let

$$g = \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}, \quad A = \begin{pmatrix} 1_2 & 0 \\ M_0 & 1_2 \end{pmatrix}$$

with M_0 in $\mathbb{C}^{(2,2)}$. Then g lies in $\mathrm{Spin}^1(V_0)$ exactly when ${}^t\bar{A}HA = H$ or ${}^t\bar{M}_0 + M_0 = 0$. Set $M_0 = iE\,{}^tA_0 E^{-1} = i\,{}^sA_0$, $A_0 = \begin{pmatrix} a^0 + a^3 & a^1 - ia^2 \\ a^1 + ia^2 & a^0 - a^3 \end{pmatrix}$; the condition on M_0 becomes ${}^t\bar{A}_0 = A_0$. If we expand the right side of $X' = AX\,{}^tA$ or

$$\begin{pmatrix} (-y' + z')E & iX_0'E \\ -{}^t(iX_0'E) & (y' + z')E \end{pmatrix}$$

$$= \begin{pmatrix} 1_2 & 0 \\ iE\,{}^tA_0E^{-1} & 1_2 \end{pmatrix} \begin{pmatrix} (-y + z)E & iX_0E \\ -{}^t(iX_0E) & (y + z)E \end{pmatrix} {}^t\begin{pmatrix} 1_2 & 0 \\ iE\,{}^tA_0E^{-1} & 1_2 \end{pmatrix}$$

we obtain

$$\begin{pmatrix} (-y + z)E & i(X_0 + (-y + z)A_0)E \\ -{}^t(i(X_0 + (-y + z)A_0)E) & * \end{pmatrix}$$

where the starred entry is

$$(y + z)E - E\,{}^tA_0E^{-1}X_0E - E\,{}^tX_0E^{-1}A_0E - (-y + z)E\,{}^tA_0E^{-1}A_0E$$

Now, $E\,{}^tA_0E^{-1}A_0 = {}^sA_0A_0 = (\det A_0)1_2 = -(aa)1_2$, where $aa = -(a^0)^2 + (a^1)^2 + (a^2)^2 + (a^3)^2$, and, by polarization, $E\,{}^tA_0E^{-1}X_0 + E\,{}^tX_0E^{-1}A_0 = -2(ax)1_2$, where $ax = -a^0x^0 + a^1x^1 + a^2x^2 + a^3x^3$. The starred entry is $((y + z) + 2ax + aa(-y + z))E$. Thus

$$X_0' = X_0 + A_0(-y + z)$$

$$-y' + z' = -y + z$$

$$y' + z' = y + z + 2ax + aa(-y + z)$$

or

$$y' = ax + (1 - \tfrac{1}{2}aa)y + \tfrac{1}{2}aaz$$

$$z' = ax - \tfrac{1}{2}aay + (1 + \tfrac{1}{2}aa)z$$

Hence

$$p(g)=\left(\begin{array}{c|cc} & -a^0 & a^0 \\ & -a^1 & a^1 \\ 1_4 & -a^2 & a^2 \\ & -a^3 & a^3 \\ \hline -a^0a^1a^2a^3 & 1-\frac{1}{2}aa & \frac{1}{2}aa \\ -a^0a^1a^2a^3 & -\frac{1}{2}aa & 1+\frac{1}{2}aa \end{array}\right)$$

in $O(V_0)$. From Section 2.8, g in $\mathrm{Spin}(V_0)$ describes the translation $x'^i = x^i + a^i$ of Ψ.

In the same manner, one ascertains that

$$g=\begin{pmatrix} A & 0 \\ 0 & \epsilon {}^tA^{-1}\end{pmatrix}, \qquad A=\begin{pmatrix} \epsilon a^{-1}1_2 & 0 \\ 0 & a1_2\end{pmatrix}, \qquad \begin{array}{l} a \neq 0 \text{ real} \\ \epsilon = \pm 1 \end{array}$$

in $\mathrm{Spin}^+(V_0)$ has image

$$p(g)=\left(\begin{array}{c|c} 1_4 & 0 \\ \hline 0 & \begin{array}{cc} \frac{1}{2}(\lambda+1/\lambda) & \frac{1}{2}(\lambda-1/\lambda) \\ \frac{1}{2}(\lambda-1/\lambda) & \frac{1}{2}(\lambda+1/\lambda) \end{array}\end{array}\right)$$

in $O(V_0)$, where $\lambda = \epsilon a^2$. From Section 2.8, g in $\mathrm{Spin}(V_0)$ describes the homothety $x'^i = \lambda x^i$ of Ψ.

Finally, from Section 6.2,

$$e_+=\left(\begin{array}{cc|cc} 0 & & -E & 0 \\ & & 0 & E \\ \hline -E & 0 & & \\ 0 & E & & 0 \end{array}\right)$$

in $\mathrm{Spin}(V_0)$ has image

$$p(e_+)=\left(\begin{array}{c|cc} 1_4 & 0 & \\ \hline 0 & -1 & 0 \\ & 0 & 1 \end{array}\right)$$

in $O(V_0)$. By Section 2.8, e_+ describes inversion in the unit sphere centered at the origin in Ψ.

6.7. The subspace of V_0 consisting of $x=\begin{pmatrix} 0 & X \\ -{}^aX & 0\end{pmatrix}$, $X=\begin{pmatrix} 0 & iX_0E \\ -{}^t(iX_0E) & 0\end{pmatrix}$, ${}^t\bar{X}_0 = X_0$, is isomorphic to W_0 of Section 3.7, for $(x|x) = -\mathrm{pf}X = \det(iX_0E) =$

$-\det X_0$. The same assertion applies to the subspace of V consisting of x with X_0 in $\mathbb{C}^{(2,2)}$ and W of Sections 3.5 ff. Such an x can be written

$$x = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{0} & 0 & X_0(iE) \\ & & {}^sX_0(iE) & 0 \\ \hline 0 & (iE)^{-1}X_0 & \multicolumn{2}{c}{0} \\ (iE)^{-1\,s}X_0 & 0 & & \end{array}\right)$$

$$= F^{-1}\left(\begin{array}{cc|cc} 0 & X_0 & \multicolumn{2}{c}{0} \\ {}^sX_0 & 0 & & \\ \hline \multicolumn{2}{c|}{0} & 0 & X_0 \\ & & {}^sX_0 & 0 \end{array}\right)F, \qquad F = \left(\begin{array}{cc|cc} 1_2 & 0 & iE & 0 \\ 0 & 1_2 & 0 & iE \\ \hline -1_2 & 0 & iE & 0 \\ 0 & 1_2 & 0 & -iE \end{array}\right)$$

Hence, the latter subspace is the image of W under the inclusion

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \to F^{-1}\left(\begin{array}{cc|cc} A & B & \multicolumn{2}{c}{0} \\ C & D & & \\ \hline \multicolumn{2}{c|}{0} & A & B \\ & & C & D \end{array}\right)F$$

$$= \left(\begin{array}{cc|cc} A & 0 & 0 & B(iE) \\ 0 & D & C(iE) & 0 \\ \hline 0 & (iE)^{-1}B & (iE)^{-1}A(iE) & 0 \\ (iE)^{-1}C & 0 & 0 & (iE)^{-1}D(iE) \end{array}\right)$$

of $\mathbb{C}^{(4,4)}$ in $\mathbb{C}^{(8,8)}$, $A, \ldots, D$ in $\mathbb{C}^{(2,2)}$. This is just the extension of the inclusion of W in V to the inclusion of $C(W) = \mathbb{C}^{(4,4)}$ in $C(V) = \mathbb{C}^{(8,8)}$.

The considerations of Section 3.3 and 3.2 apply, and the inclusion above sends $\mathrm{Spin}(W_0)$ into $\mathrm{Spin}(V_0)$. Indeed, from the descriptions of these groups in Sections 3.8 and 6.3, one obtains the inclusion on the various components to be

$$\begin{pmatrix} A_0 & 0 \\ 0 & {}^{st}\bar{A}_0 \end{pmatrix}, \quad \det A_0 = +1 \to \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}, \quad A = \begin{pmatrix} A_0 & 0 \\ 0 & {}^t\bar{A}_0^{-1} \end{pmatrix}, \qquad \epsilon = +1$$

$$\begin{pmatrix} A_0 & 0 \\ 0 & {}^{st}\bar{A}_0 \end{pmatrix}, \quad \det A_0 = -1 \to \begin{pmatrix} A & 0 \\ 0 & -{}^tA^{-1} \end{pmatrix}, \quad A = \begin{pmatrix} A_0 & 0 \\ 0 & -{}^t\bar{A}_0^{-1} \end{pmatrix}, \qquad \epsilon = -1$$

$$\begin{pmatrix} 0 & B_0 \\ {}^{st}\bar{B}_0 & 0 \end{pmatrix}, \quad \det B_0 = +1 \to \begin{pmatrix} 0 & B \\ -{}^tB^{-1} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & iB_0E \\ i{}^t\bar{B}_0^{-1}E & 0 \end{pmatrix}, \qquad \epsilon = -1$$

$$\begin{pmatrix} 0 & B_0 \\ {}^{st}\bar{B}_0 & 0 \end{pmatrix}, \quad \det B_0 = -1 \to \begin{pmatrix} 0 & B \\ {}^tB^{-1} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & iB_0E \\ -i{}^t\bar{B}_0^{-1}E & 0 \end{pmatrix}, \qquad \epsilon = +1$$

Here we employ ${}^sA_0 = A_0^{-1}$ or $-A_0^{-1}$ as $\det A_0 = +1$ or -1 to make the necessary calculations. Note that the elements $1, e_0, e_1e_2e_3, e_0e_1e_2e_3$ of 3.8 which represent the components of Spin(W_0) are sent to those of Section 6.3 which represent the components of Spin(V_0).

From Section 3.2, this inclusion of Spin(W_0) in Spin(V_0) corresponds to the inclusion $v \to \begin{pmatrix} v & 0 \\ 0 & 1_2 \end{pmatrix}$ of $O(W_0)$ in $O(V_0)$. One component of the image of Spin(W_0) lies in each component of Spin(V_0), likewise for $O(W_0)$ and $O(V_0)$. The correspondence respects the components.

6.8. Conformal transformations of Ψ constitute the group $PO(V_0) = O(V_0)/\{\pm 1_6\}$. Rotations or Lorentz transformations of Minkowski space constitute the group $O(W_0)$, are conformal transformations, and are obtained from the inclusion $v \to \begin{pmatrix} v & 0 \\ 0 & 1_2 \end{pmatrix} \bmod \{\pm 1_6\}$ of $O(W_0)$ in $PO(V_0)$. The homomorphism $g \to p(g) \bmod \{\pm 1_6\}$ maps Spin(V_0) onto $PO(V_0)$ with kernel $\{\pm 1_8, \pm z\}$, $z = \begin{pmatrix} -i1_4 & 0 \\ 0 & i1_4 \end{pmatrix}$. From Section 6.7, the image of Spin(W_0) in Spin(V_0) meets this kernel in exactly $\{\pm 1_8\}$, so this homomorphism restricts to the image of Spin(W_0) as the homomorphism p of Spin(W_0) onto $O(W_0)$. Hence, as in 3.4, the inclusion of Spin(W_0) in Spin(V_0) corresponds to the inclusion of $O(W_0)$ in $PO(V_0)$.

$$\begin{array}{ccc} \mathrm{Spin}(W_0) & \xrightarrow{\text{incl.}} & \mathrm{Spin}(V_0) \\ {\scriptstyle 2\text{ to }1}\downarrow & & \downarrow{\scriptstyle 4\text{ to }1} \\ O(W_0) & \xrightarrow{\text{incl.}} & PO(V_0) \end{array}$$

The correspondence respects components.

Thus the geometry of Minkowski space with group $O(W_0)$ extends to the conformal geometry of its compactification Ψ with group $PO(V_0)$ as a natural consequence of the inclusion of Spin(W_0) in Spin(V_0).

References

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